An Unified Theory of Collective Behavior of Cooperative Multi-agent System

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Background

- The consensus of multi-agent systems has been intensively discussed by many authors. For example
  - De Groot(1974) and Berger(1981) for statistical model.
  - Tsitsklis(1984,1986) for optimal control systems.
  - Heppner(1990) for nonlinear Leader-Flollow systems.
  - Fax and Murray (2001) Olfati-Saber and Murray (2004) , Lin(2005) for linear continuous systems.
  - Moreau(2005) for linear system under time-invariant network.
  - ……

Our study

- Our study focus on the general model of cooperative system.
- “The consensus problem” of multi-agent cooperative systems will be replaced by “the collective behavior” of multi-agent cooperative systems.

Multi-Agent System

- **A** is a set of agents. a_i (i = 1, ... , n) is an agent, whose state is x_i , $x_i \in \mathbf{R}^m$ (m=1,2,3,…).
- The set of a_i's neighbors is denoted by $N(t, a_i)$, which consists of all agents acting on a_i at time t.
- The neighbor set of agents defines a directed network

$$\mathbf{G}(t) = < \mathbf{A}, \mathcal{E}(t) >$$

where

$$\mathcal{E}(\mathrm{t}) = \left\{ (a_i, a_j) \middle| a_i \in \mathbf{A}, a_j \in N(t, a_i) \right\}$$

- Multi-Agent System

$$\dot{x}_i(t) = f_i(x_i, x_{j_1}, \ldots, x_{j_k}) \qquad x_{j_1}, \ldots, x_{j_k} \in N(t, a_i) \qquad a_i \in \mathbf{A} \quad (1)$$

Affine model

- A general model

$$\dot{x}_i(t) = \sum_{a_j \in N(t,a_i)} f_{ij}(x_i, x_j) \qquad a_i \in \mathbf{A} \qquad (2)$$

- Galilean relativity
  - A dynamical system is of Galilean relativity if all experimental results obtained in different observation systems are the same.
  - Model (2) is Galilean relativity if and only if it is affine; and it must take the form of (3) (Jin and Zheng (2011,[1])).

- Affine model

$$\dot{x}_i(t) = f_i = \sum_{a_j \in N(t,a_i)} q_{ij}\left(\left\|x_i - x_j\right\|\right)\vec{\mathbf{r}}_{ij} = q_{ij}\left(\left\|x_i - x_j\right\|\right)\frac{x_i - x_j}{\left\|x_i - x_j\right\|} \qquad a_i \in \mathbf{A} \qquad (3)$$

where $q_{ij}(.) \in$ R is scalar function.

Cooperative Systems

An affine multi-agent dynamic system (3)

$$\dot{x}_i(t) = f_i = \sum_{a_j \in N(t,a_i)} -g_{ij}\left(\left\|x_i - x_j\right\|\right)\vec{\mathbf{r}}_{ij} \qquad a_i \in \mathbf{A} \qquad (4)$$

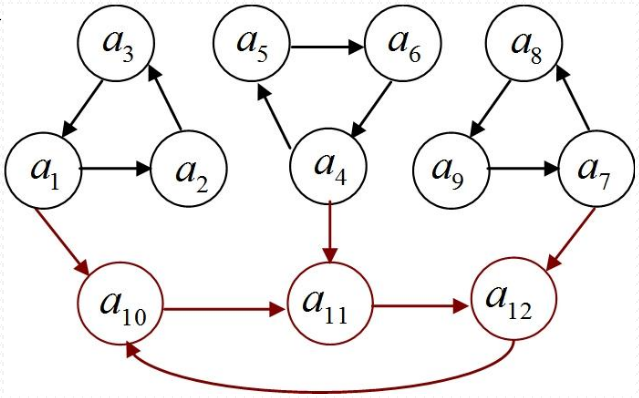
is a cooperative system if

$$g_{ij}\left(\left\|x_i - x_j\right\|\right)\begin{cases} = 0 & \left\|x_i - x_j\right\| = 0 \\ > 0 & \left\|x_i - x_j\right\| > 0 \end{cases}$$

g_{ij} is the strength acting on a_i by a_j .

Structure of Directed Networks

- **G** is a directed network, then it can be decomposed into several strong component (maximal strongly connected sub-graph of **G**). Some of them are independent and some are non-independent.



- $\mathbf{G}_1$ =< $\{a_1, a_2, a_3\}$, $\{(a_1, a_3), (a_2, a_1), (a_3, a_2)\}$ >,$\mathbf{G}_2$,$\mathbf{G}_3$ are in-dependent strong components.

- $\mathbf{G_4}$=< $\{a_{10}, a_{11}, a_{12}\}$, $\{(a_{10}, a_{12}), (a_{11}, a_{10}), (a_{12}, a_{11})\}$ > is a non-independent strong component.

Basic sets

- $\mathbf{A}_s \subset \mathbf{A}$. If induced sub-graph

$$\mathbf{G}_s = \langle \mathbf{A}_s, \mathcal{E} \cap (\mathbf{A}_s \times \mathbf{A}_s) \rangle$$

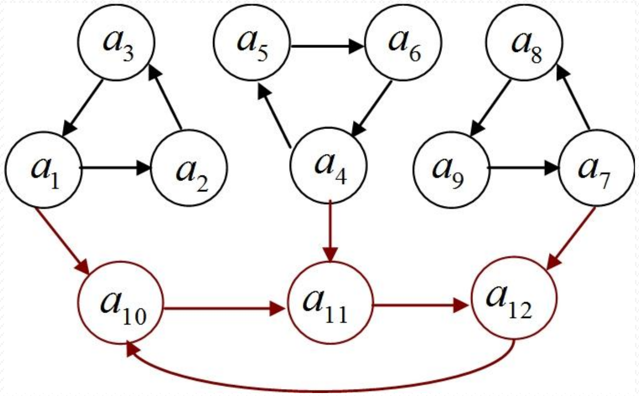
is a strong component of **G**, then $\mathbf{A}_s$ is a basic set of **A**.

- $\mathbf{G}_s$ is independent, then $\mathbf{A}_s$ is an independent basic set; $\mathbf{G}_s$ is non-independent, then $\mathbf{A}_s$ is a non-independent basic set.
- The independent basic sets:

$\mathbf{A}_1 = \{a_1, a_2, a_3\}, \mathbf{A}_2 = \{a_4, a_5, a_6\}, \mathbf{A}_3 = \{a_7, a_8, a_9\}$

- The non-independent basic sets:

$\mathbf{A}_4 = \{a_{10}, a_{11}, a_{12}\}$



Assumptions

1. $\mathbf{G}(t)$ is piecewise constant

$$\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_k, \ldots$$

where $\mathbf{G}(t) = \mathbf{G}_k =< \mathbf{A}, \mathcal{E}_k >$, when $t \in [t_k, t_{k+1})$. $\mathbf{G}_k$ is a time-invariant network.

2. There exists $l > 0$ such that

$$t_{k+1} - t_k > l\ .$$

3. g_{ij} are continuous and smooth

   The cooperative system (4) is continuous and smooth over every time interval $[t_k, t_{k+1})$, and then the trajectories of system (4) are continuous and piecewise smooth.

Adjoint Graph

- Definition

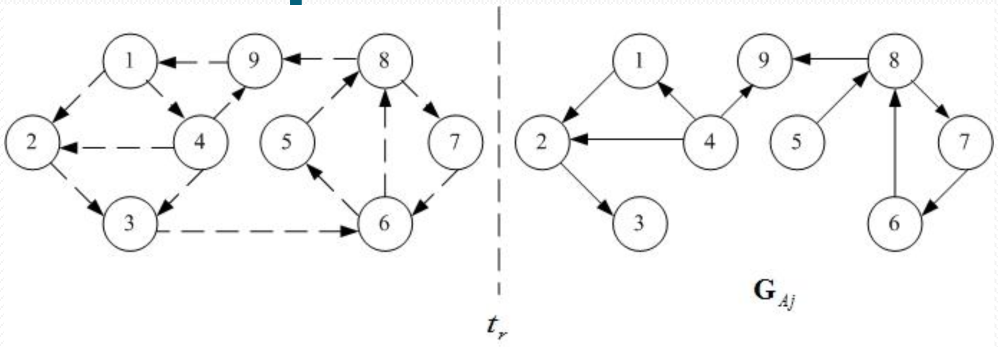
$\mathbf{G}_{aj} =< \mathbf{A}, \mathcal{E}_{aj} >$ is called an adjoint graph of a time-varying network $\mathbf{G}(t)$ if

$$\mathcal{E}_{aj} = \bigcap_{k=1}^{\infty} \bigcup_{s=k}^{\infty} \mathcal{E}_s$$

I.e. for any $t \in [0,\infty)$, there exist $t_{\mathrm{k}} \geq \mathrm{t}$, $(a_i , a_j) \in \mathcal{E}_{\mathrm{k}}$, then $(a_i , a_j)_{aj} \in \mathcal{E}_{aj}$, which is called an adjoint edge of $\mathbf{G}(t)$. $\mathcal{E}_{aj}$ is the set of adjoint edges.

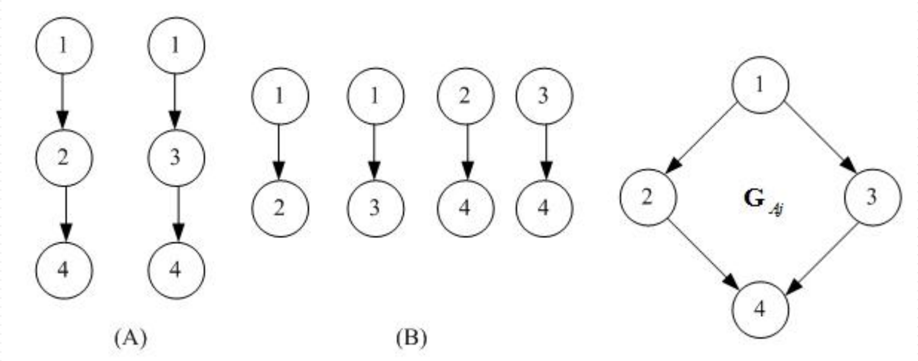
- Based on adjoint graph $\mathbf{G}_{aj}$ of $\mathbf{G}(t)$, one may study the system (4) as a system under time-invariant network.

Adjoint Graph



- The left part consists of the edges which may appear in the graph during the time interval $[0, t_r)$, may not. But some of them will disappear over the time interval $[t_r, \infty)$ for some $t_r \gg 1$. These edges may have no effect upon the collective behavior of system. The right part only includes the edges which will persistently appear over the time interval $[0,\infty)$ although the edges may intermittently appear. These edges are adopted by the adjoint graph $\mathbf{G}_{aj}$ of $\mathbf{G}(t)$.

Adjoint Graph



- A describes a periodic switch network. The connectivity of agents is described in the way repeating from one to the other when $t \in [0,\infty)$.
- B is another periodic switch network repeating from the first case to the fourth case.
- Both networks, A and B, have same adjoint graph $\mathbf{G}_{aj}$.

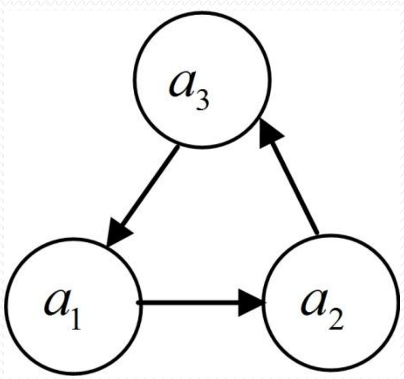
Collective Behavior

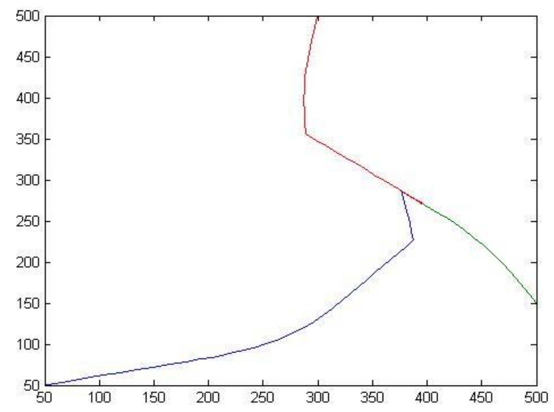
For cooperative systems (4), which satisfy mentioned Assumptions, the paper by Jin and Zheng(2011(2)) gives the following results. η

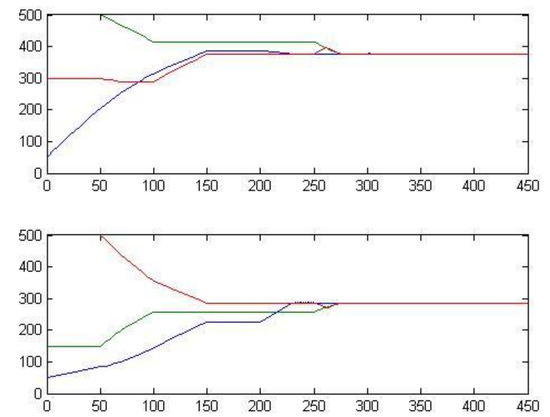
- **Theorem 1**: All of agents in an independent basic set $\mathbf{A}_s$ of $\mathbf{A}$ on $\mathbf{G}_{aj}$ of $\mathbf{G}(t)$ will tend to a fixed state, i.e. there exists

η

$$\lim_{t\to\infty} x_i = \eta_s \in \mathbf{R}^m, \quad a_i \in \mathbf{A}_s$$



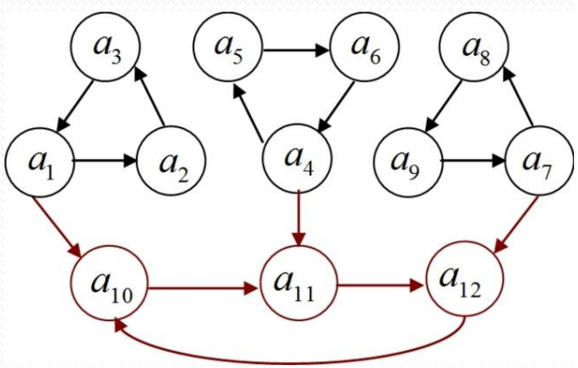


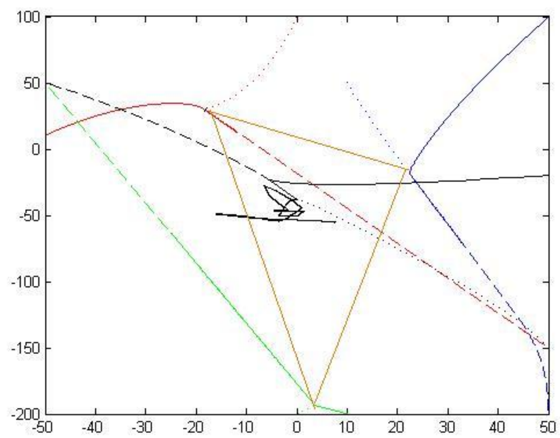


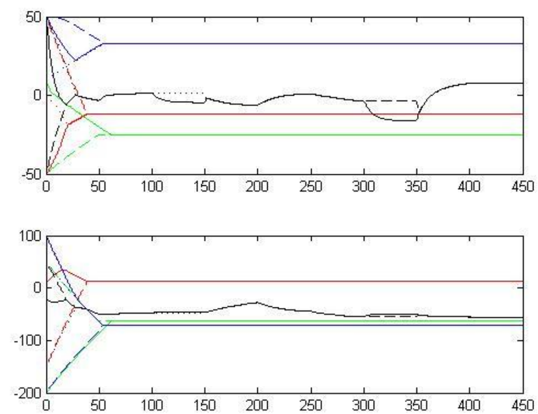
Collective Behavior

- **Theorem 2:** $\mathbf{\Lambda} = \{c_1, \cdot\cdot\cdot, c_p\}$ is the set of the consensus states of all independent basic sets $\mathbf{A}_s$ ($s = 1, ..., p$) of $\mathbf{A}$ on $\mathbf{G}_{aj}$ of $\mathbf{G}(t)$. $\Xi \subset \mathbf{R}^m$ is the smallest convex set such that $\Lambda \subset \Xi$. Then for any $a_i \in \mathbf{A}$, as $t \rightarrow \infty$,

$$x_i(t) \in \Xi$$

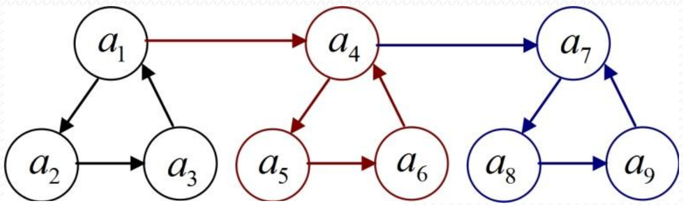


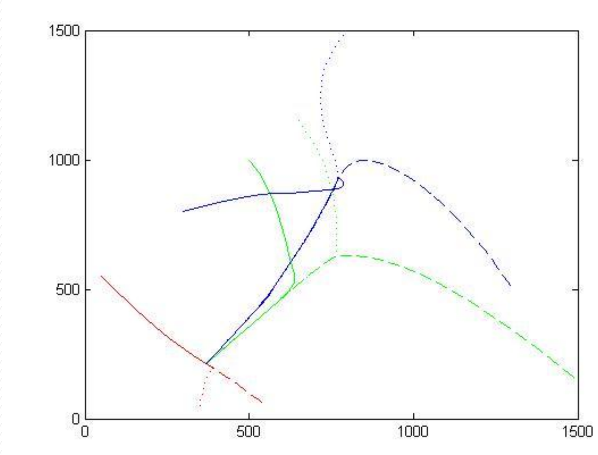


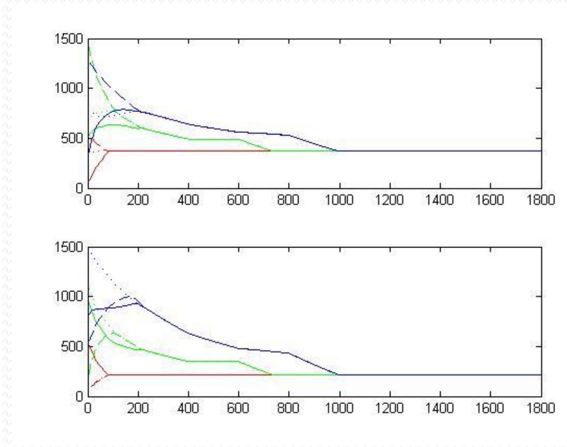


Collective Behavior

- **Corollary 1:** All states of the cooperative system (4) tend to consensus if and only if there is only one independent basic set of **A** on $\mathbf{G}_{aj}$ of $\mathbf{G}(t)$.







- **Corollary 2:** $\mathbf{G}_{aj}$ of $\mathbf{G}(t)$ is strongly connected, then all states of the cooperative system (4) tend to consensus.

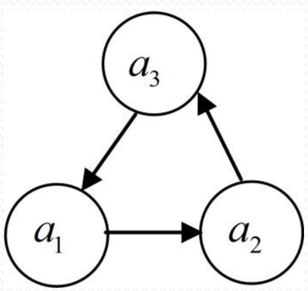
Conclusion

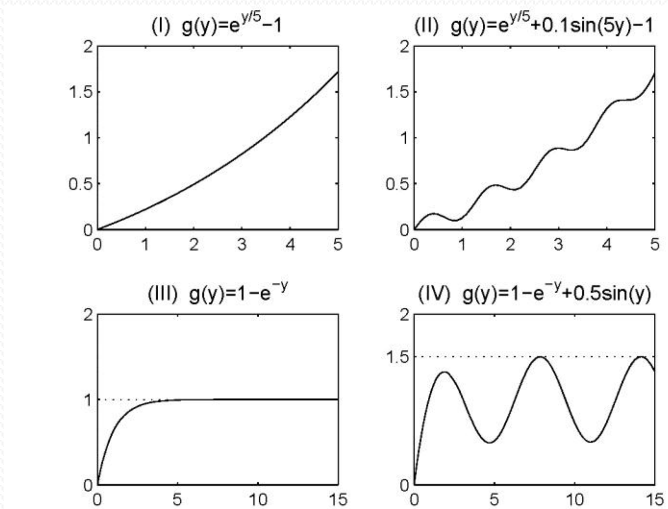
The collective behavior of the cooperative system (4) which satisfy Assumptions is determined by the topology of adjoint graph $\mathbf{G}_{aj}$ of $\mathbf{G}(t)$.

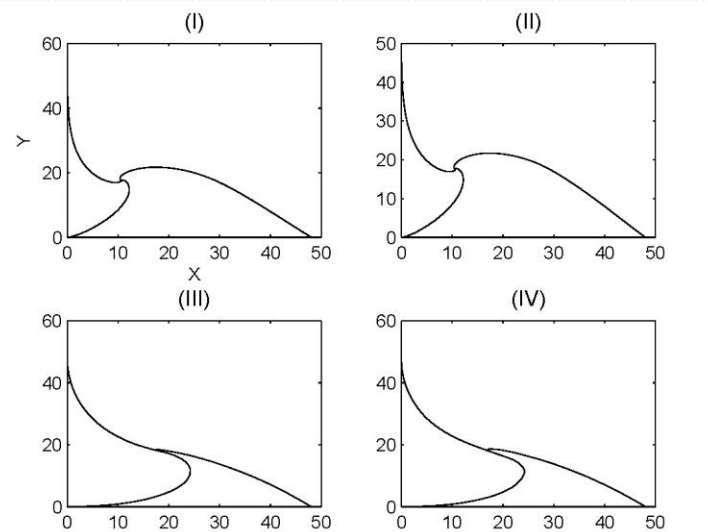
Time-Invariant Network

G(t) = **G** is a time-invariant network, then the adjoint graph $\mathbf{G}_{aj}$ of **G**(t) is itself ($\mathbf{G}_{aj}$ = **G**).

- (Jin and Zheng 2011(1)): **G**(t) = **G**, then all of agents in an independent basic set $\mathbf{A}_s$ of A on **G** will tend to a fixed state.

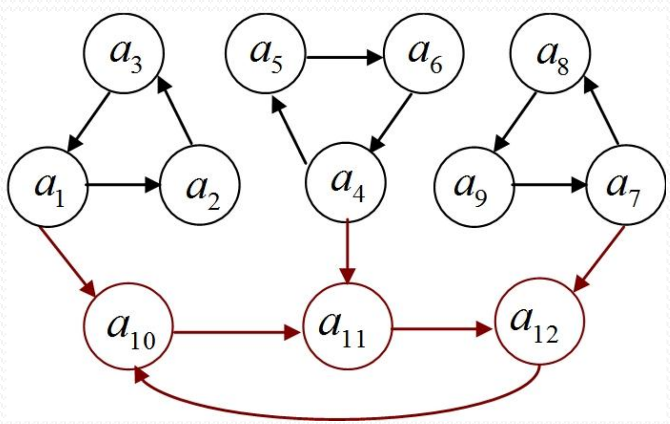


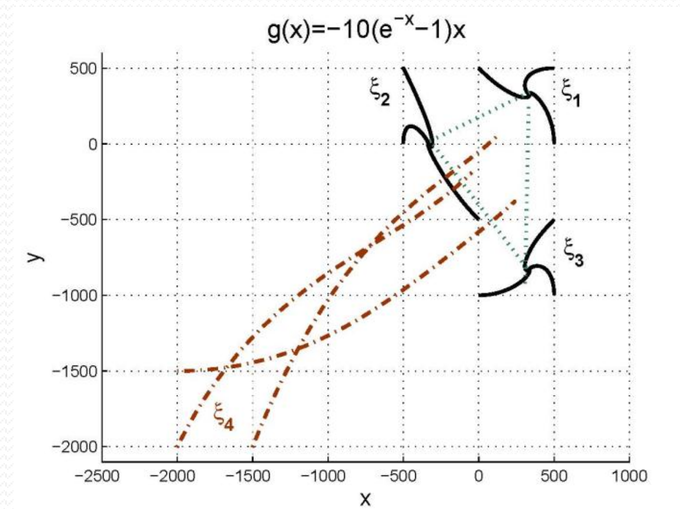




Time-Invariant Network

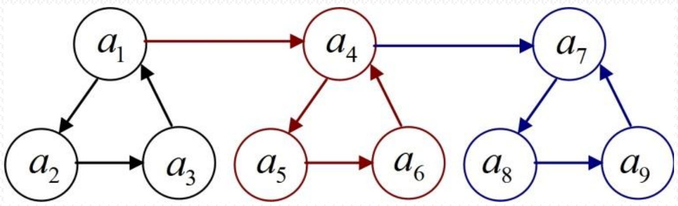
- (Jin and Zheng 2011(1)): $\mathbf{G}(t) = \mathbf{G}$. $\Lambda = \{c_1, \cdots, c_p\}$ is the set of the consensus states of all independent basic sets $\mathbf{A}_s$ ($s = 1, \ldots, p$) of $\mathbf{A}$ on $\mathbf{G}$. $\Xi \subset \mathbf{R}^m$ is the smallest convex set such that $\Lambda \subset \Xi$. Then for any $a_i \in \mathbf{A}$, as $t \to \infty$, $x_i(t) \in \Xi$.

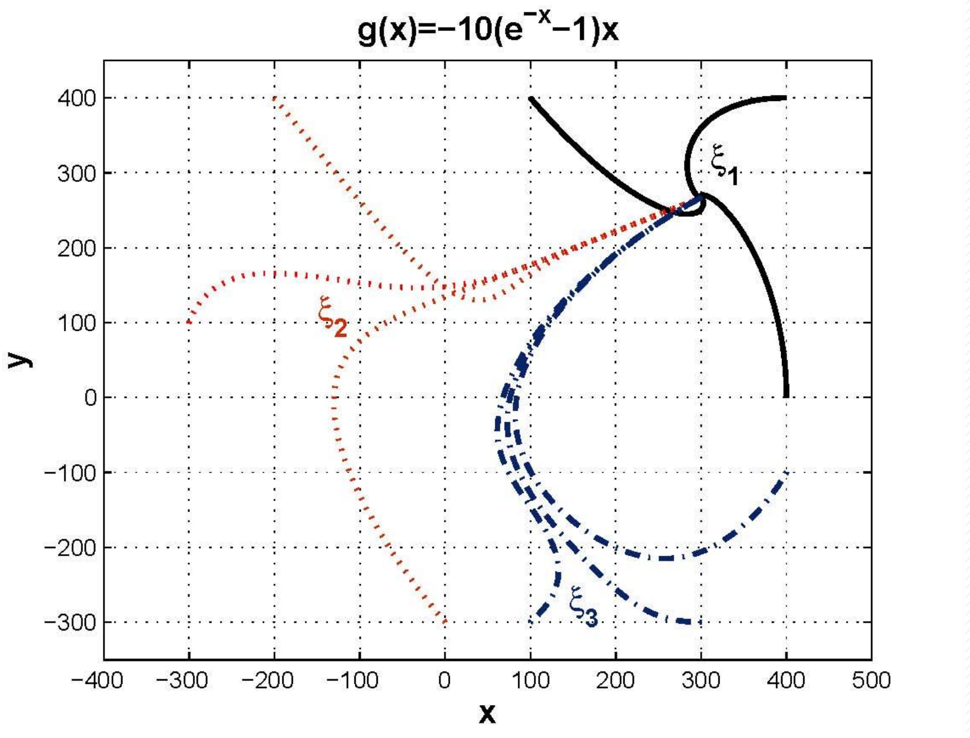




Time-Invariant Network

- (Jin and Zheng 2011(1)): **G**(t) = **G**. All states of the cooperative system (4) tend to consensus if and only if there is only one independent basic set of **A** on **G**.





Linear Cooperative System

- If for any given i , j , there exists a constant $\gamma_{ij} \in \mathbf{R}$:

$$\gamma_{ij} = \frac{g_{ij}\left(\left\|x_i - x_j\right\|\right)}{\left\|x_i - x_j\right\|} < 0$$

then (4) is a linear multi-agent cooperative system.

$$x_i = \sum_{a_j \in N(t,a_i)} \gamma_{ij}(x_i - x_j) \qquad (5)$$

The linear cooperative system (5) is a special type of affine cooperative system (1). Therefore, the problem of collective behavior of linear cooperative system (5) under time-varying network is solved by this study and it covers the results of preliminary studies under time-invariant network given by Fax, Olfati-Saber and Murray (2001,2004), Lin(2005) , Jin and Zheng(2009).

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