On Game-based Control of Logical Dynamic Systems

– A Semi-tensor Product Formulation

Daizhan Cheng

Institute of Systems Science
Academy of Mathematics and Systems Science
Chinese Academy of Sciences

Workshop on Flocking, Swarming, and Control of Multi-agent Systems
Shanghai, May 10, 2012

Outline of Presentation

I. Game-based Control Systems

☞ Background

- Control of Boolean (Genetic Regulatory) Network [1]

  [1] A. Datta, A. Choudhary, M.L. Bittner, E.R. Dougherty, External control in Markovian genetic regulatory networks: the imperfect information case, *Bioinformatics*, Vol. 20, No. 6, 924-930, 2004.

- Flight Control (Missile Defence System) [2]

  [2] C.J. Tomlin, J. Lygeros, S.S. Sastry, A game theoretic approach to controller design for hybrid systems, *Proc. of IEEE*, Vol. 88, No. 7, 2000.

- Control of Power Systems (Power Grid) [3]

  [3] W.W. Weaver, P.T. Krein, Game-theoretic control of small-scale power systems, IEEE Trans. Power Delivery, Vol. 24, No. 3, 1560-1567, 2009.

☞ Static Game

Notations:

- $\mathcal{D}_k = \{1, 2, \cdots, k\}$;
- $\Delta_k = \{\delta_k^i | i = 1, 2, \cdots, k\}$, where $\delta_k^i = \mathrm{Col}_i(I_k)$.

Definition 1.1

A static game G consists of three ingredients:

(i) n players, named $p_1, \cdots, p_n$;

(ii) each player p_i has k_i possible actions, denoted by $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdot n$;

(iii) n payoff functions for n players respectively as

$$c_j(x_1 = i_1, \cdots, x_n = i_n) = c^j_{i_1\, i_2\, \cdots\, i_n}, \quad j = 1, \cdots, n. \tag{1}$$

Definition 1.2

In a static game G,

(1) A set of actions $s = (x_1, \cdots, x_n)$, is a strategy (or strategy profile) of G. The set of strategies is denoted by S.

(2) A strategy $\{x_j^*\}$ is a Nash equilibrium if

$$c_j(x_1^*, \cdots, , x_j^*, \cdots, x_n^*) \geq c_j(x_1^*, \cdots, x_j, \cdots, x_n^*) \quad j = 1, \cdots, n. \tag{2}$$

Example 1.3

Consider a game G with two players: P_1 and P_2:

- Actions of P_1: $\mathcal{D}_2 = \{1, 2\}$;
- Actions of P_2: $\mathcal{D}_3 = \{1, 2, 3\}$.

Table 1: Payoff bi-matrix

$P_1 \backslash P_2$	1	2	3
1	2, 1	$\underline{3}$, $\underline{2}$	6, 1
2	1, 6	2, 3	5, 5

Nash Equilibrium is $(1, 2)$.

☞ Antagonistic Games

Definition 1.4

A (static) game G is called an antagonistic game, if

1. there is a partition of players as
$$P_1 \cup P_2 = \{p_1, p_2, \cdots, p_n\};$$
2. there are two payoff functions for two groups:
$$\begin{aligned} P_1 &: \quad c_1(x_1, \cdots, x_n); \\ P_2 &: \quad c_2(x_1, \cdots, x_n). \end{aligned}$$

e.g., Missiles vs Anti-missile Missiles.

The game is equivalent to two-player one.

☞ Dynamic Games

Assumptions:

(i) infinitely repeated:

$$G \to G_\infty$$

(ii) evolutive strategy:

$$x_i(t+1) = f_i(x_1(t), \cdots, x_n(t)), \quad i = 1, \cdots, n.$$

Payoffs:

$$\begin{aligned} J_i &= \overline{\lim}_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} c_i(x_1(t), \cdots, x_n(t)), \\ &\quad i = 1, \cdots, n. \end{aligned} \tag{3}$$

☞ From Dynamic Game to Control System

Assume (i)antagonistic; (ii) evolutive strategies. Then from each side P_i, ($i = 1, 2$) we have control systems as

$$x(t+1) = f(x(t), u(t)); \quad x \in \mathcal{D}_p;\ u \in \mathcal{D}_q. \tag{4}$$

$$\max_{u(i),\ i=0,1,\cdots} J, \tag{5}$$

where

$$J = \overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=1}^{T} c(x(t), u(t)).$$

II. Semi-tensor Product of Matrices

☞ Definition of STP

$A_{m\times n} \times B_{p\times q} =?$

Definition 2.1

Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$. Denote

$$t := \mathrm{lcm}(n, p).$$

Then we define the semi-tensor product (STP) of A and B as

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{(mt/n)\times(qt/p)}. \tag{6}$$

References

[4] D. Cheng, H. Qi, *Semi-tensor Product of Matrices and Its Applications*, 2nd Ed., Science Press, Beijing, 2011. (In Chinese)

[5] D. Cheng, H. Qi, Z. Li, *Analysis and Control of Boolean Networks - A Semi-tensor Product Approach*, Springer, London, 2011.

[6] D. Cheng, H. Qi, Y. Zhao, *An Introduction to Semi-tensor Product of Matrices and Its Applications*, World Scientific, Singapore, 2012. (to appear)

☞ Some Basic Comments

- When $n = p$, $A \ltimes B = AB$. So the STP is a generalization of conventional matrix product.
- When $n = rp$, denote it by $A \succ_r B$;
  when $rn = p$, denote it by $A \prec_r B$.
  These two cases are called the multi-dimensional case, which is particularly important in applications.
- STP keeps almost all the major properties of the conventional matrix product unchanged.

☞ Examples

Example 2.2

1. Let $X = \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then

$$X \ltimes Y = \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 3 & -1 \end{bmatrix} \cdot 2 = \begin{bmatrix} 7 & 0 \end{bmatrix}.$$

2. Let $X = \begin{bmatrix} -1 & 2 & 1 & -1 & 2 & 3 \end{bmatrix}^T$ and $Y = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix}$. Then

$$X \ltimes Y = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 2 + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot (-2) = \begin{bmatrix} -3 \\ -6 \end{bmatrix}.$$

Example 2.2 (Continued)

3. Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}.$$

Then

$$A \ltimes B = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & -3 & -5 \\ 4 & 7 & -5 & -8 \\ 5 & 2 & -7 & -4 \end{bmatrix}.$$

☞ Matrix Expression of Logical Functions

Vector Form of Logical Variables

Definition 2.3

(i) Assume $x \in \mathcal{D}_k$, its vector form is defined as $\vec{x} = \delta_k^x$.

(ii) $L \in \mathcal{M}_{k\times n}$ is called a logical matrix, if $Col(L) \in \Delta_k$, that is,

$$L = \left[\delta_k^{i_1}, \delta_k^{i_2}, \cdots, \delta_k^{i_n}\right].$$

Briefly,

$$L = \delta_k\left[i_1, i_2, \cdots, i_n\right].$$

(iii) The set of $k \times n$ logical matrices is denoted by $\mathcal{L}_{k\times n}$.

☞ Matrix Expression of Logical Functions (continued)

Theorem 2.4

Let $y \in \mathcal{D}_{k_0}$ and $x_i \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$, and

$$y = f(x_1, \cdots, x_n). \tag{7}$$

Then there exists an unique matrix $M_f \in \mathcal{L}_{k_0 \times k}$ $(k = \prod_{i=1}^{n} k_i)$ such that in vector form

$$y = M_f \ltimes_{i=1}^{n} x_i. \tag{8}$$

M_f is called the structure matrix of f, and (8) is the algebraic form of (7).

☞ Algebraic Form of Logical Control Systems

Algebraic form of (4).

$$x(t+1) = Lu(t)x(t); \quad x \in \mathcal{D}_p;\ u \in \mathcal{D}_q, \tag{9}$$

where

$$L \in \mathcal{L}_{p \times pq}.$$

III. Optimal Control

☞ Key Results

Consider (4)(or (9) with performance criterion (5):

Theorem 3.1

(1) The best strategy is state-control periodic.

(2) The best strategy $(u^*(t))$ satisfies

$$u^*(t+1) = g(x(t), u(t)) = L_g u(t) x(t), \tag{10}$$

where $L_g \in \mathcal{L}_{p\times pq}$.

☞ Method 1:

Proposition 3.2

Cycle:

$$(x_1, u_1) \to (x_2, u_2) \to \cdots \to (x_k, u_k) = (x_1, u_1)$$

is called a simple cycle, if

$$x_i \neq x_j, \quad 1 \leq i < j < k.$$

Proposition 3.3

For any cycle C there exists a simple cycle C_s, such that the average payoff

$$\bar{c}(C_s) \geq \bar{c}(C).$$

☞ Method 1 (continued):

(1) Find set of simple cycles;

(ii) Find

$$\bar{c}(C_s^*) = \max_{C_s} \bar{c}(C_s).$$

(ii) Decomposing C_s^* yields a best trajectory and a best control sequence.

☞ Method 2:

Definition 3.4

Let $A, B \in \mathcal{L}_{m\times n}$. The Hamming distance is defined as

$$d_H(A,B) = \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{i,j} - b_{i,j}|.$$

Find L^* (via hill climbing) such that

$$J(L^*) = \max_{L_g \in \mathcal{L}_{p\times pq}} J(L_g).$$

[7] Y. Zhao, Z. Li, D. Cheng, Optimal control of logical control networks *IEEE Trans. Aut. Contr.*, vol.56, no. 8, pp. 1766-1776.

[8] D. Cheng, Y. Zhao, Y. Mu, Strategy optimization with its application to dynamic games, *Proc. 49th IEEE CDC*, 5822-5827, Atlanta, 2010.

☞ Discount Factor

$$J = \sum_{t=1}^{\infty} \lambda^t c(x(t), u(t)), \tag{11}$$

where $0 < \lambda < 1$ is the discount factor.

Theorem 3.5

Consider (4)(or (9) with performance criterion (11). If the optimal control exists, then Theorem 3.1 holds.

[9] D. Cheng, Y. Zhao, J. Liu, Optimal Control of Finite-valued Networks *Proc. WCICA'12*. (to appear)

IV. Mixed Strategy Systems

Assume P_1 use mixed strategy. Then (9) becomes

$$x(t+1) = L_i u(t) x(t), \quad i = 1, 2, \cdots, s,$$

with

$$P(L = L_i) = p_i, \text{ and } \sum_{i=1}^{s} p_i = 1.$$

Now if we consider x as the distribution of states, we still have (9) with

$$L = \sum_{i=1}^{s} p_i L_i.$$

☞ Finite Horizon Case

$$J = E\left[\sum_{k=0}^{m-1} c_k(u(k), x(k)) + c_m(x(m)) \Big| x(0)\right] \qquad (12)$$

Theorem 4.1

(Dynamic Programming) Let J^* be the optimal value of (12). Then

$$J^*(x_0) = J_0(x_0),$$

where J_0 comes from Algorithm 4.2.

[10] A. Datta, A. Choudhary, M.L. Bittner, E.R. Dougherty, Exernal control in Markovian genetic regulatory networks, *Machine Learing*, Vol. 52, 169-191, 2003.

☞ Finite Horizon Case (continued)

Algorithm 4.2

$$J_m(x(m)) = c_m(x(m)); \tag{13}$$

$$\begin{aligned} J_k(x(k)) \quad &= \quad \max_{u(k)} E\left[c_k(u(k), x(k)) + J_{k+1}(x(k+1))\right], \\ & \quad\quad k = m-1, m-2, \cdots, 1, 0. \end{aligned} \tag{14}$$

☞ STP Expression(continued)

Let

$$c_k(\delta_q^i, \delta_p^j) := \mu_{i,j}^k, \quad i = 1, \cdots, q;\ j = 1, \cdots, p.$$

We construct

$$C_k := (\mu_{i,j}^k) \in \mathcal{M}_{q\times p}.$$

Then

$$c_k(u, x) = u^T C_k x.$$

Notations:

(i) $u^i(k)$: the control for $x(k) = \delta_p^i$, $i = 1, \cdots, p$;

(ii) Consider J_k as a vector with

$$J_k = [J_k(\delta_p^1), \cdots, J_k(\delta_p^p)]^T.$$

☞ STP Expression(continued)

$$J_k = \max_{u^1,\cdots,u^p} \left\{ \begin{bmatrix} \langle u^1, \mathrm{Col}_1(C_k) \rangle \\ \vdots \\ \langle u^p, \mathrm{Col}_p(C_k) \rangle \end{bmatrix} + \begin{bmatrix} u^1 \\ \vdots \\ u^p \end{bmatrix} L^T J_{k+1} \right\}. \qquad (15)$$

Set

$$V^i := \mathrm{Col}_i(C_k) + L^T J_{k+1}, \quad i = 1, \cdots, p.$$

Then

Proposition 4.3

J_k and the optimal control can be calculated as

$$\begin{aligned} J_k(i) &= \max_j V^i_j := V^i_{j^*}, \\ u^i(k) &= \delta_q^{j^*}. \end{aligned} \qquad (16)$$

☞ Infinite Horizon Case

$$J = \lim_{m\to\infty} E\left[\sum_{k=0}^{m-1} \lambda^k c(u(k), x(k)))\Big| x(0)\right]. \tag{17}$$

Define a mapping $T : \mathbb{R}^p \to \mathbb{R}^p$ as

$$(TJ)_i = \max_u \left[c(u, \delta_p^i) + \lambda u^T L^T J\right], \quad i = 1, \cdots, p. \tag{18}$$

[11] P. Pal, A. Datta, E.R. Dougherty, Optimal infinite-holozon control for probabilistic Boolean networks, *IEEE Trans. Signal Processing*, Vol. 54, No. 6, 2375-2387, 2006.

☞ Infinite Horizon Case(continued)

Theorem 4.4

For any bounded J, the optimal payoff satisfies

$$J^* = \lim_{m \to \infty} T^m J. \tag{19}$$

Theorem 4.5

The optimal payoff is the unique solution of

$$J^* = TJ^*. \tag{20}$$

☞ Infinite Horizon Case(continued)

Proposition 4.6

The optimal payoff satisfies

$$\max \left[\mathrm{Col}_i(C) + \lambda L^T J^* \right] = J_i^*, \quad i = 1, \cdots, p. \tag{21}$$

V. Evolutionary Game on Multi-agent Systems

Model:

- Agents: $\{1, 2, \cdots, n\}$
- Neighborhoods: $\{N_i | i = 1, \cdots, n\}$;
- Player i is gambling with each Player $j \in N_i$.

[12] C. Hauert, M. Doebeli, Spatial structure oftern inhibites the evolution of coopration in the snowdrift game, *Nature*, Vol. 438, 643-646, 2004.

[13] C. F.C. Santos, M.D. Santos, J.M. Pacheco, Social diversity promotes the emergence of cooperation in public goods games, *Nature*, Vol. 454, 213-216, 2008.

Evolutive Strategies:

$$\begin{aligned} x_i(t+1) &= L_i(t)x_i(t)x_j(t), \quad \forall j \in N_i, \\ & i = 1, 2, \cdots, n. \end{aligned} \tag{22}$$

payoff:

$$\begin{aligned} c_i(t) &= \sum_{j \in N_i} H_{i,j} x_i(t) x_j(t), \\ & i = 1, 2, \cdots, n. \end{aligned} \tag{23}$$

☞ Question?

How to choose the evolutive matrix $L_i(t)$ to realize:

- emergence of cooperation?
- coherence?
- other global properties?
- $\cdots$

☞ Update $L_i(t)$:

Best Substitution:

$$L_i(t) = L_{j^*}(t-1); \tag{24}$$

where

$$j^* = \text{argmax}_{j \in \{N_i(t) \cup i\}} c_j(t).$$

Weighted Average Substitution:

$$L_i(t) = \sum_{k \in N_i \cup i} \frac{c_i(t)}{c_0(t)} L_k(t-1); \tag{25}$$

where

$$c_0(t) = \sum_{j \in N_i(t) \cup i} c_j(t).$$

VI. Conclusion

(1) Game-based control systems have logical type;

(ii) Semi-tensor product can convert logical type dynamics into algebraic type dynamics;

(iii) Algebraic form may provide an easy way to solve the optimization problem.

(iv) Optimal controls lead to Nash equilibrium.

(v) Evolutionary games can be described precisely via semi-tensor product.

Thank you for your attention!

Question?