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Set Coordination of multi-agent systems

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Outline

- Introduction
- Preliminaries
- Containment
- Set optimization
- Conclusions

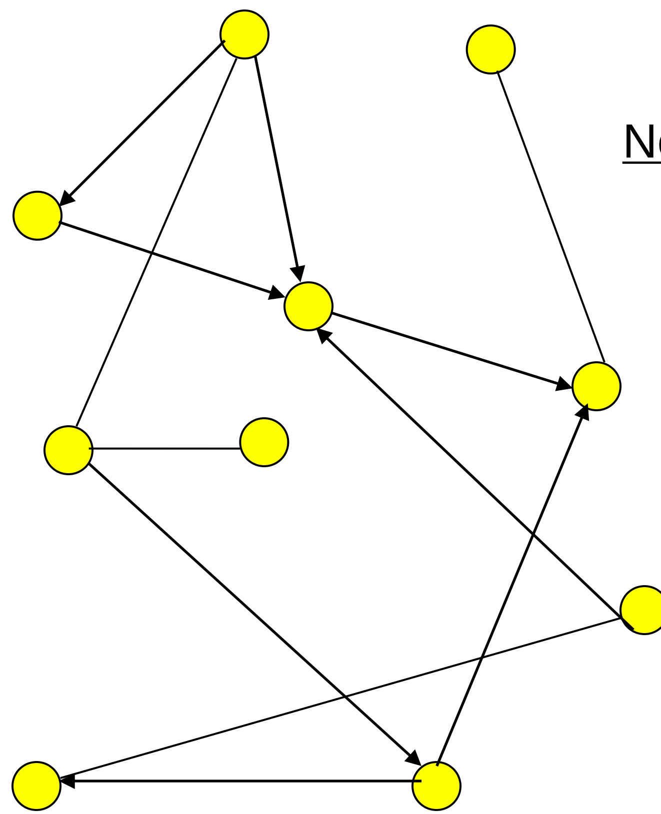
1. Multi-agent Systems

- Agent: subsystem with sensors and actuators
- In Nature: particles, neural network, ecological system, metabolic system…
- In Society: urban development, opinion dynamics, transportation network…
- In Engineering: power grid, wired/wireless communication, sensor network, robotic network, software system…

Interaction → Graph

Graph for the interaction topology of agents → Laplacian or stochastic matrices

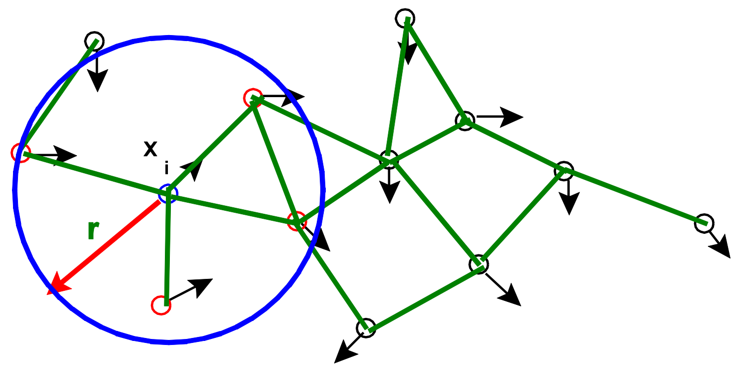
•Directed graph: directed link (without self-loop)
•Undirected graph → bidirectional graph (undirected connections but maybe different weights)



Node (agent)

Link (information flow): directed or undirected

Consensus: a simple case



Neighbor Graph

Agent dynamics: $dx_i/dt = u_i \;\; i = 1,\ldots 2$
Leader (or desired position): x_0

Neighbor-based communication (N_i : the neighbor set of agent i)

Distributed control: $u_i = \Sigma_j \, (x_j - x_i), \; j \in N_i$

Multi-agent consensus (agreement, synchronization):

- **Leader-following:** $x_i - x_0 \rightarrow 0$
- **Leaderless:** $x_i - x_j \rightarrow 0$

Multi-agent Coordination

- Coordination Problems:
  - Consensus (agreement, synchronization): a basic problem
  - Rendezvous, flocking, swarm
  - Formation, sweeping …
- Coordination → connectivity
  - Interconnection topology: information flow
  - Switching topologies: joint connection
  - Leader-following or leaderless

Distributed analysis & control

Interconnection Topology:

Graph theory

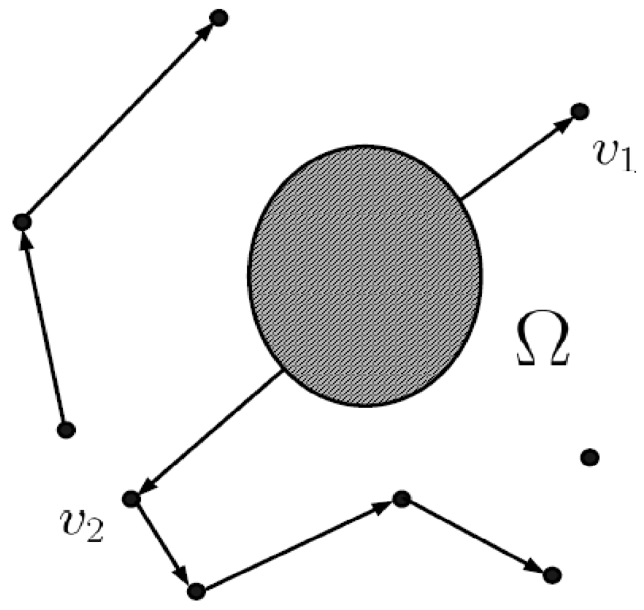
Collective Dynamics:

Stability theory

Hierarchy and Interaction

Set coordination

- What happens if a set Ω gets involved with the coordination of a group of agents?
- How to give distributed algorithms for set tracking or set optimization?
- What if there are multiple leaders? …



Motivation

Migration or swarm to food source or nest site

Couzin, et al., **Leadership by Numbers**

Nature, vol 433., Feb. 2005

Results: The effectiveness of the leaders (informed agents)

Motivation 2

Agents (sheep) are contained in a region spanned by a group of leaders (sheepdogs) → **containment control**:

- Multiple leaders: form a moving polytope (convex target set)
- Multiple followers (agents): aim at reaching the convex leader-set.

Motivation 3

- Practical target set (due to inaccuracy in measurement, uncertainty in environment…) for coordination or decision making
- Distributed set optimization, optimal consensus within a convex set

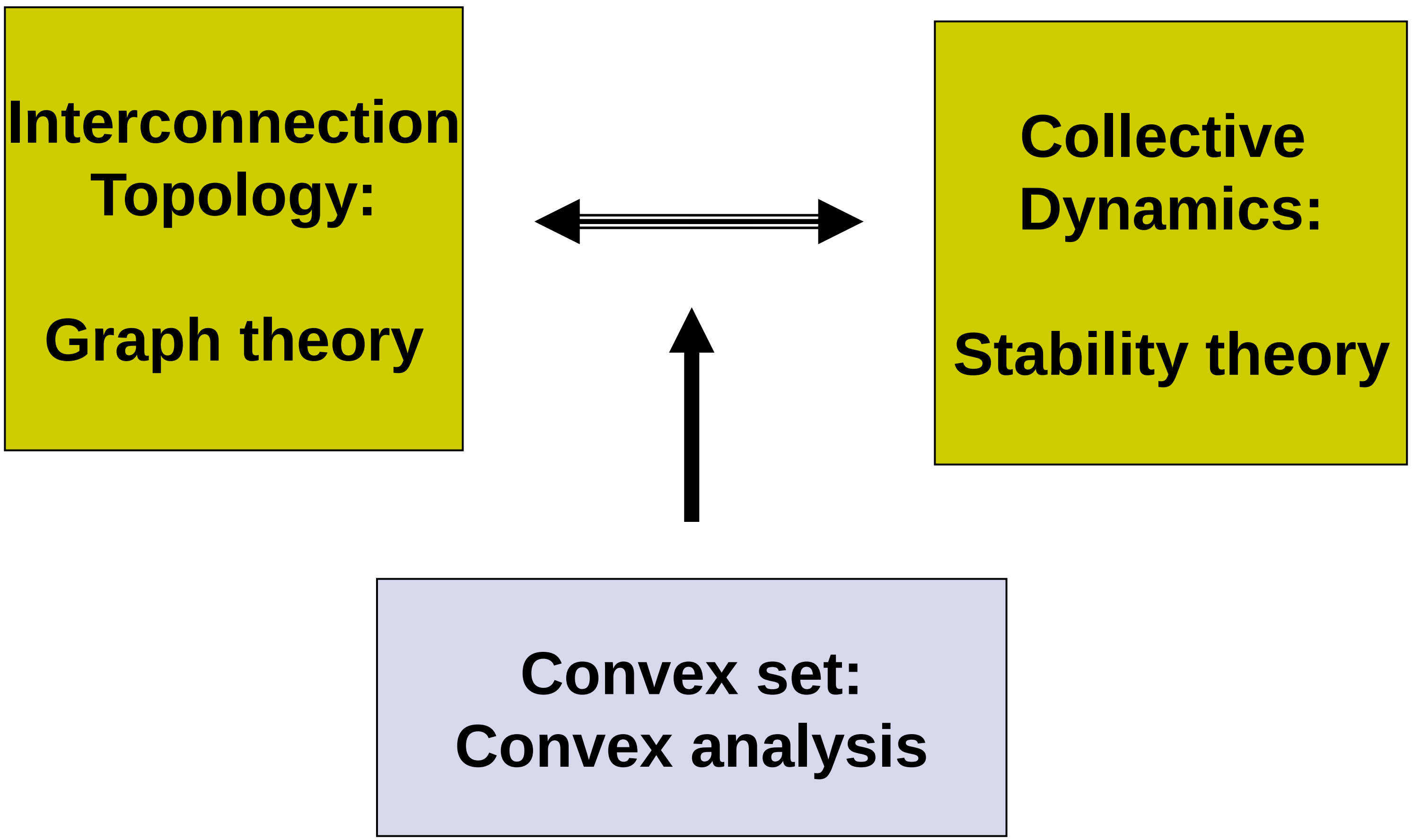
Set coordination problems

- Flocking to a given position → Drive a group of agents into a target convex set Ω
- Leader-following → multi-leader following → containment control: contain the agents in a convex set spanned by leaders
- Multi-agent consensus → set optimization: consensus together with convex optimization

Difficulties

- Centralized design → Distributed design: neighbor-based rule, not completely connected
- Conventional stability → set stability (non-smoothness)
- Switching interaction topology (non-smoothness): common Lyapunov function

Distributed algorithm with set



Related results

- Stationary set: Couzin, et al. Nature, 2005; Lin, et al, IEEE TAC'05 (a segment); Shi, Hong, Automatica'09 (any convex set); …
- Moving-leader containment: Ji et al, IEEE TAC'07; Ren et al, IEEE TAC'10; Shi, Hong, et al IEEE TAC'12; Lou, Hong, Automatica'12
- Convex optimization with distributed consensus: Johansson et al, IEEE CDC'08; Nedic et al, IEEE TAC'09, 10; Shi, Johansson, Hong, (IEEE TAC conditionally accepted 2012), Lou et al, WCICA'12.

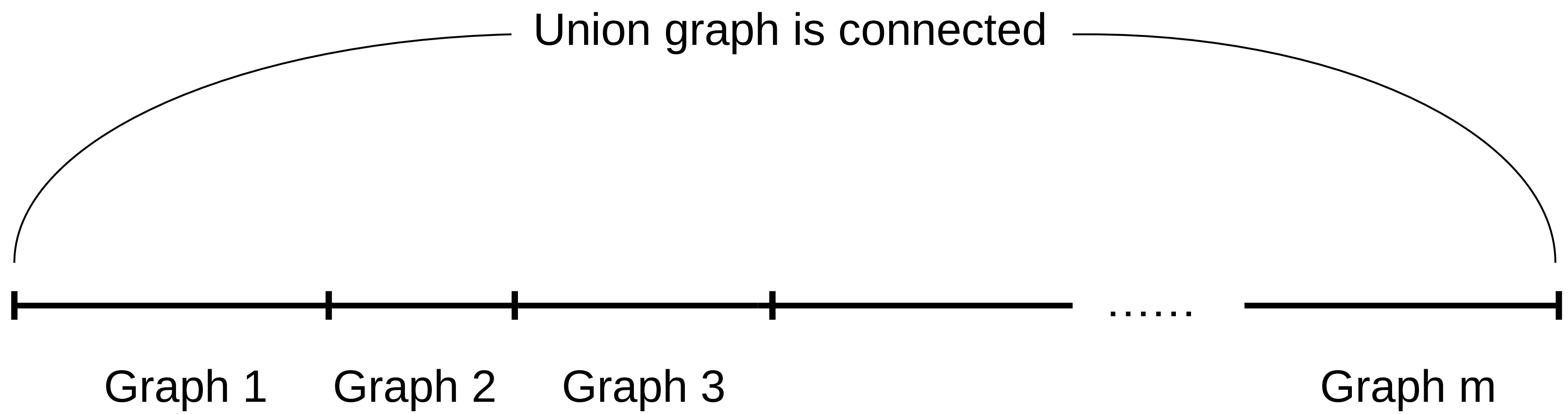
2. Preliminaries

Graph theory: the interaction topology of agents (which is important especially in the homogenous agent case) → algebraic graph theory: Laplacian or stochastic matrices.

- Directed graph (without self-loop)
- Undirected graph → bidirectional graph (undirected connections but weights may be different)

Connectivity

- Joint connection: union graph in $[t, \infty)$ is connected for any t: a necessary condition
- Uniform joint connection: $\exists T$, union graph on $[t, t+T]$ is connected for any t.



Convex set

- K is a convex set if, for $0 < \lambda < 1$

$$(1-\lambda)x + \lambda y \in K \qquad x \in K, y \in K$$

- $d(x,K)$: distance between set K and x

$$\|x\|_K \triangleq \inf\{\|x-y\| | y \in K\}$$

Dini derivative

- Dini derivative: for a continuous function h

$$D^+h(t) = \limsup_{s\rightarrow 0+} \frac{h(t+s)-h(t)}{s}$$

- Dini derivative for a switching function V:

  if $V(t,x) \;=\; \max_{i=1,2,\cdots,n} V_i(t,x)$

$$D^+V(t,x(t)) = \max_{i\in\mathcal{I}(t)} \dot{V}_i(t,x(t))$$

$$\mathcal{I}(t) = \{i \in \{1,2,\cdots,n\} \;:\; V(t,x(t)) = V_i(t,x(t))\}$$

Input-to-state Stability (ISS)

- ISS proposed by Sontag for nonlinear control systems, widely used in stabilization & robust control (for interconnected system)
- $dx/dt=f(x,u)$, x: state; u: input
- The system is ISS if $||x(t)|| \leq \beta(||x(0)||, t) + \gamma(||u||_{\infty})$, with *K*-function γ and *KL*-function β
- ISS → bounded input bounded output;

  ISS → asymptotic stability if no input;

Related discussion

- Integral ISS (iISS): the system is integral ISS if
  $||x(t)|| \leq \beta(||x(0)||, t) + \int^{t} \gamma(||u(s)||)ds$
- Set ISS (SISS) by Sontag: static/fixed set and system without switching
- There are few ISS results for multi-agent systems with fixed topologies (Scardovi et al, IEEE CDC'09; Tanner et al, IEEE TRA'04)

Not the set case we studied for switched multi-agent systems with moving sets: new SISS & SiISS

3. Containment

- Multiple leaders: form a moving polytope (convex target set)
- Multiple followers (agents): aim at reaching the convex target set.
- Tracking a moving set in a distributed way
- Bounded tracking error if the uncertainties are bounded

Distributed design

- Leaders: $dy_i/dt=v_i(y,t),\ i=1,\ldots,k;\ v_i$ is the velocities
- Agents: $dx_i/dt=u_i,\ i=1,\ldots,n;\ u_i$ is a neighbor-based rule in a nonlinear form with uncertainty w_i :

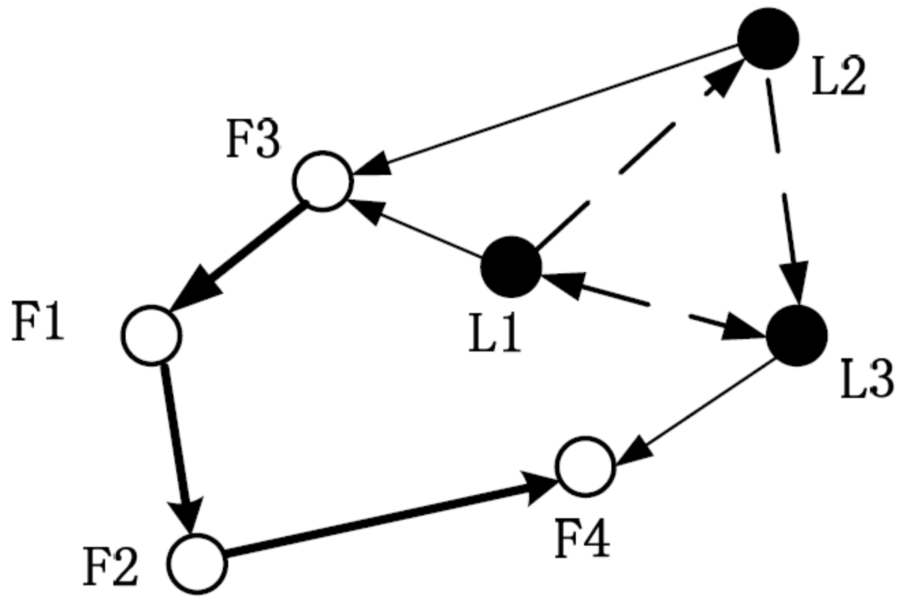
$$\sum_{j\in N_i(\sigma(t))} a_{ij}(x,y,t)(x_j-x_i)+\sum_{j\in L_i(\sigma(t))} b_{ij}(x,y,t)$$
$$\cdot(y_j-x_i)+w_i(t), \quad i=1,\cdots,n$$

- Uncertainties: leaders with unknown velocities v; agents with disturbance w

Basic setup

- The position is measurable, but the velocity is unmeasurable
- Position information of a leader is known by an agent if and only if the agent is connected to the leader: local controller only contains position information
- The interconnection topology is time-varying (widely used in multi-agent control)

Variable topology



L_i: moving leaders → span a convex set Ω

F_i: follower agents, time-varying communication

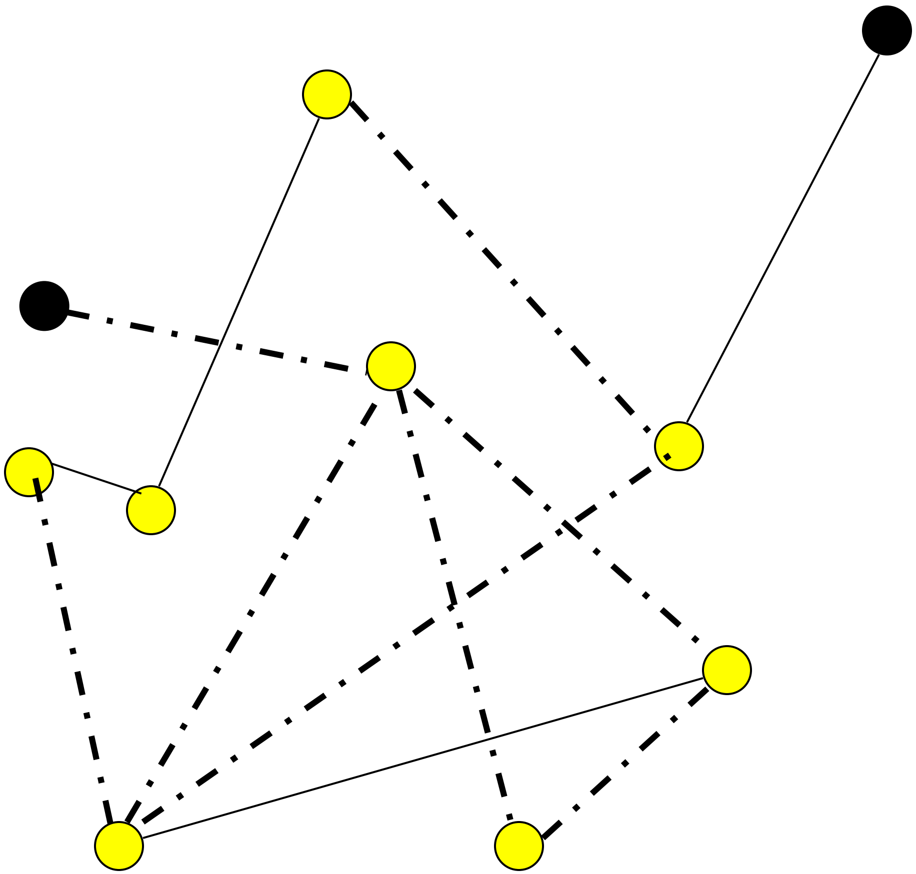
Time-varying graph → L-connected; jointly L-connected, …

L-connection

<u>L-connected</u>: any agent is accessible to a leader;

<u>Jointly L-connected (JLC)</u> : union graph of interval $[s,t]$ is L-connected;

<u>Uniformly JLC</u>: $\exists\ T$ such that the union graph of $[t,\ t+T]$ is L-connected



Set tracking and SISS

- SISS → set tracking if there is no uncertainty
- SISS with the uncertainties (v,w) as input with respect to the target set Ω
- *K*-function: γ; *KL*-function: β
- SISS: $d(x_i, \Omega) \leq \beta(d(x_i(0), \Omega), t) + \gamma(||v,w||_\infty)$
- SiISS (set integral input-to-state stability): $d(x_i, \Omega) \leq \beta(d(x_i(0), \Omega), t) + \int^t \gamma(||v,w||)ds$

Analysis

- Different from ISS, no equivalence relationship with ISS-Lyapunov function
- Constructing non-smooth Lyapunov function: $V(x)=max_i\, d(x_i, \Omega)$
- Estimating the convergence rate for SISS or SiISS by finding suitable K-function γ and KL-function β.
- Estimation is done for each interval in the case of joint L-connection

Main results: SISS

Key result 1: SISS of MAS = uniformly jointly L-connected (UJLC)

Remarks:

- SISS → bounded (uncertain) input, bounded tracking error
- Completely L-connected → SISS

Main results: SiISS

Key result 2: Uniformly jointly L-connected → SiISS of MAS

Key result 3: For bidirectional graph, SiISS = $[t,\infty)$ jointly L-connected for any t

<u>Remarks:</u>

- For our system: SISS (UJLC) → SiISS
- Other sufficient conditions: Complete or Acyclic L-connection … → SiISS

Set tracking

- Set tracking (without error): the agents enter the convex target set.
- Set tracking $\leftarrow$

1. SISS + (v,w) vanishing; or
2. SiISS + $\int^{\infty}\gamma$ bounded

Consistent with ISS in the conventional single nonlinear system.

4. Distributed set optimization

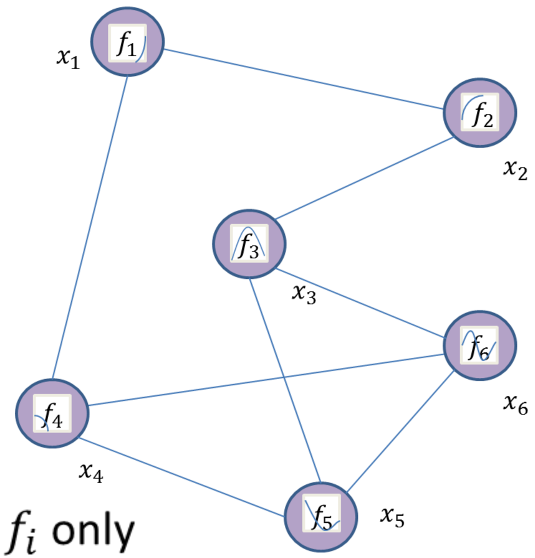
Optimization of the whole system with global cost function when each agent optimizes its own cost.

- A hot topic: convex cost functions (corresponding to convex set); local information; topology may be switching …
- Our result: weak connectivity condition, continuous-time system with a new method, optimization with consensus.

Introduction

- Distributed optimization over networks

$$\min_{z \in R^m} \quad F(z) = \sum_{i=1}^{N} f_i(z)$$



- Each node i observes its own cost function f_i only
- Information is exchanged locally over communication graphs
- All nodes target to reach the optimal solution set $X_0 = \arg\min_{z \in R^m} F(z)$

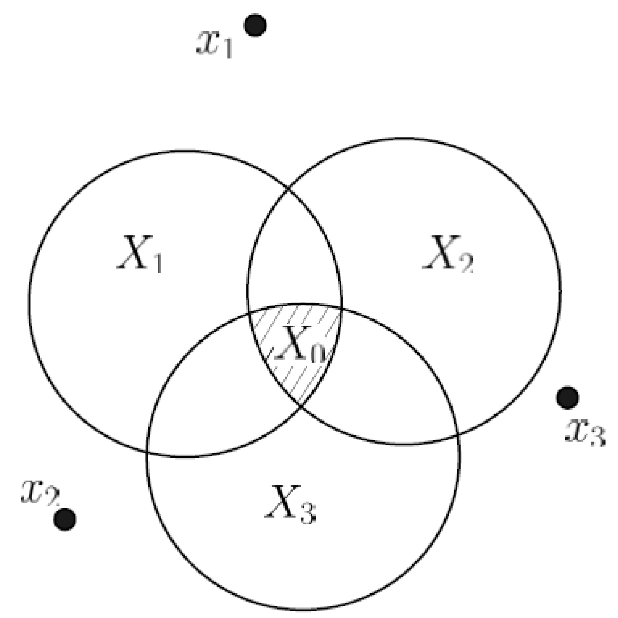
- Applications
  - Coordination control of multi-agent systems
  - Resource allocation in wireless systems
  - Opinion dynamics in social networks

Methods

- Gradient-based: manipulation of the primal variable through (sub)gradient with local (environment) information …
- Neighbor-based: optimization with local communication information
- Swarm intelligence, genetic algorithm, … (hard to be analyzed)

Our Problem

- Agent $dx_i/dt = u_i$ only knows the information of its own convex set X_i and its neighbor x_j → the agents achieve consensus within X_0 (=∩X_i, which is not empty)
- Distributed convex intersection (computer): Projected consensus algorithm (PCA) by Nedic et al 2010……



Optimal consensus

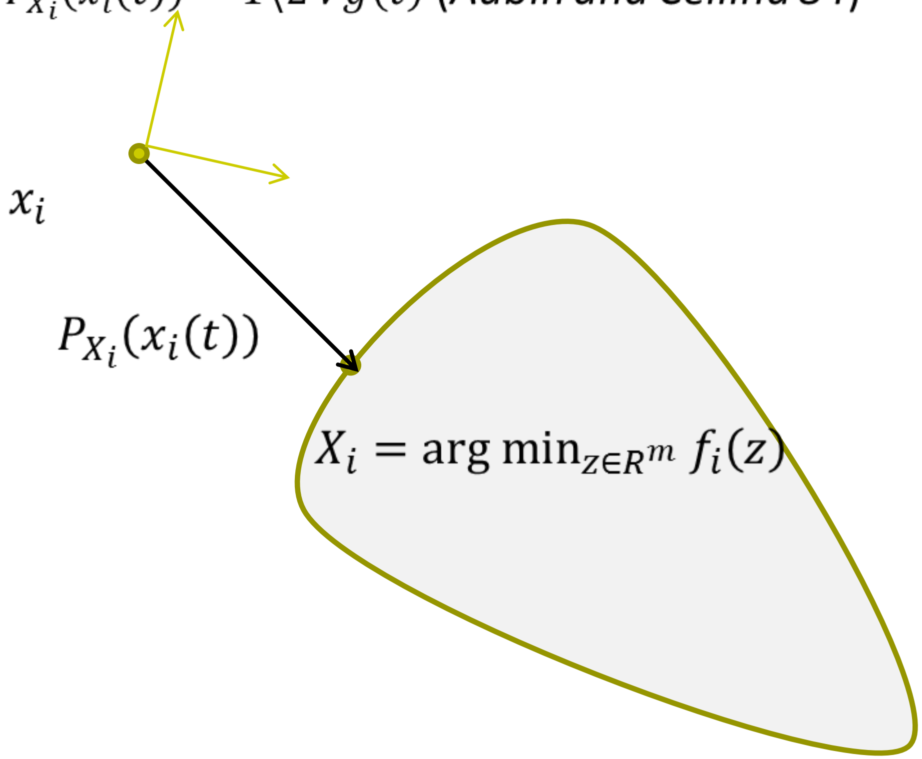
Establish relationships between connectivity conditions and optimal consensus.

<u>Basic model:</u>

- Connectivity: uniformly jointly-connected for digraph, $[t, \infty)$ jointly-connected for bidirectional (undirected) graph
- Agents: $dx_i/dt=u_i,\ i=1,\ldots,n;\ u_i$ a nonlinear local rule = subgradient for optimization + neighbor-based rule for consensus
- The intersection set $X_0=\cap X_i$ is not empty
- Dwell time

Convex projection

- ✓ Build a local coordinate frame
- ✓ Constructing a function $g(t) = |x_i\ (t)|_{X_i}{}^2$
- ✓ $x_i(t) - P_{X_i}(x_i(t)) = 1\backslash 2\ \nabla g(t)$ (*Aubin and Cellina 84*)



Distributed Control

- Control law for node i:

$$\dot{x}_i = u_i = \sum_{j \in \mathcal{N}_i(t)} a_{ij}(x,t)(x_j - x_i) + P_{X_i}(x_i) - x_i$$

- A simple combination of a consensus algorithm and a projection item
- A global coordinate system is not needed
- Each node does not need to know its state

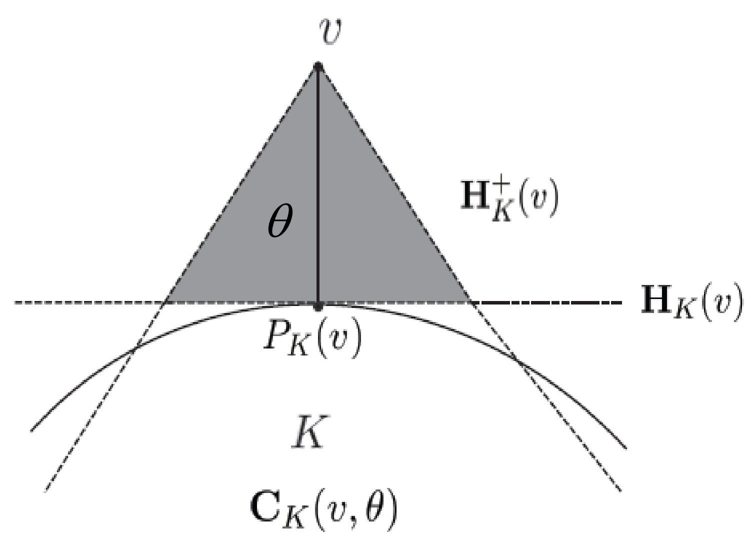
Main results

Result 1: Global optimal consensus of MAS $\leftarrow$ uniformly jointly strongly connected

Result 2: In the bidirectional case, MAS achieves global optimal consensus = $[t, \infty)$ joint connection

Approximate projection

In practice, it is hard to get accurate projection (PCA) → approximate projection with $0 \le \theta \le \theta^* < \pi/2$ (APCA):



$C_K(v, \theta)$-v is a cone

Optimal consensus: given APCA, there is $x^* \in X_0$ such that

$$\lim_{k\to\infty} x_i(k) = x^*,\ i = 1, \ldots, n$$

$$\mathscr{P}_K^{sa}(v,\theta) = \begin{cases} \mathbf{C}_K(v,\theta) \bigcap \mathbf{H}_K(v), & \textit{if } v \notin K \\ \{v\}, & \textit{if } v \in K \end{cases}$$

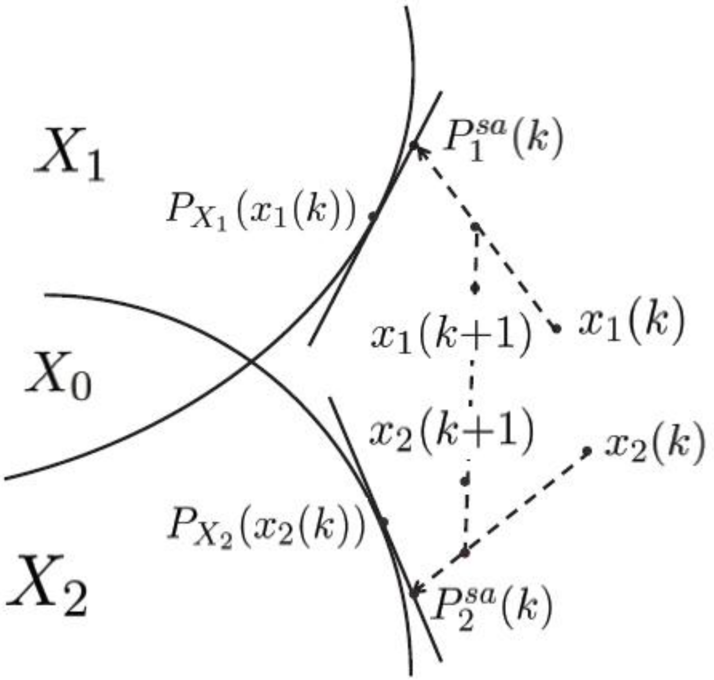
Distributed Control

Controller: $$x_i(k+1) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k)\Big((1-\alpha_{j,k})x_j(k) + \alpha_{j,k}P_j^{sa}(k)\Big)$$

where $$P_i^{sa}(k) \in \mathscr{P}_{X_i}^{sa}(x_i(k), \theta_k)$$

$$\alpha_k^- = \min_{1\le i\le n} \alpha_{i,k}$$

$$\alpha_k^+ = \max_{1\le i\le n} \alpha_{i,k}$$



Main results for APCA

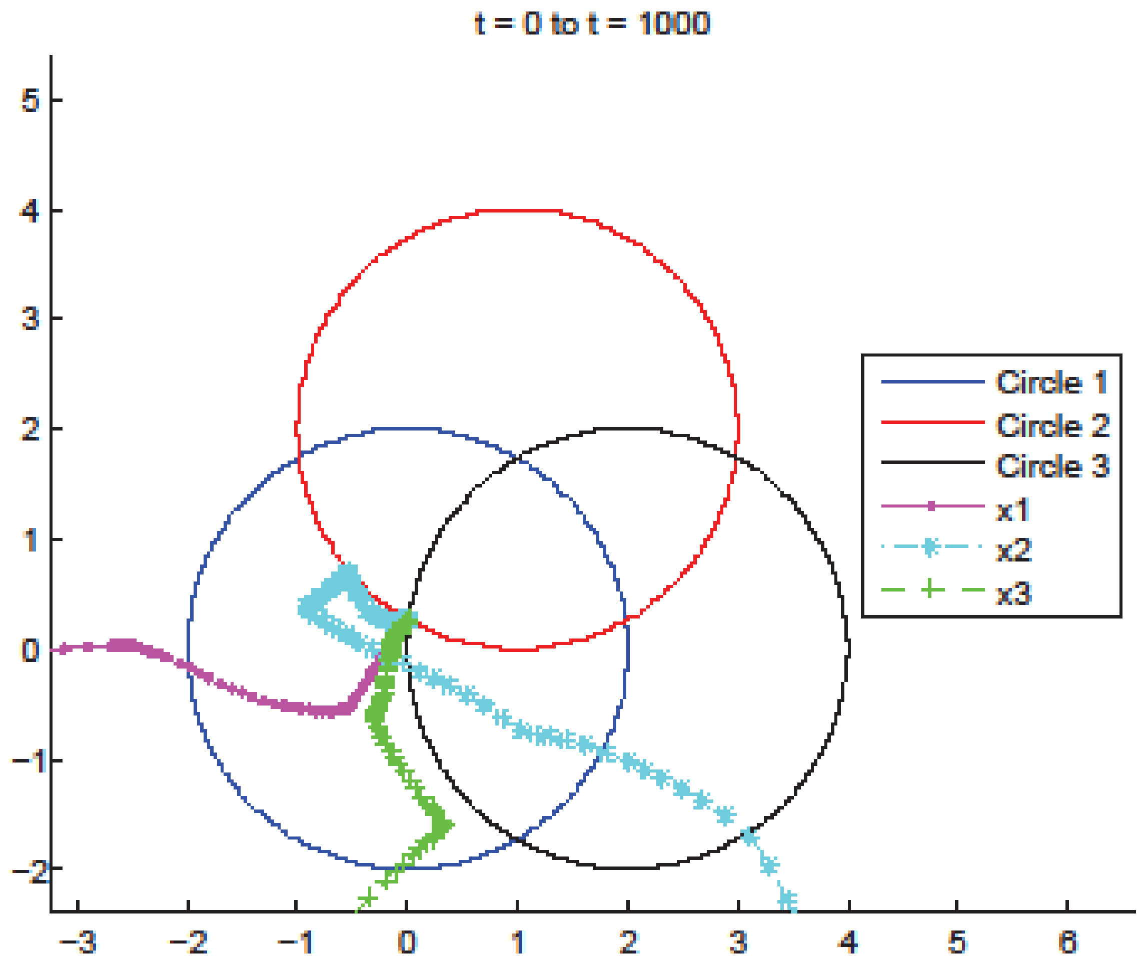
Result 1: Global optimal consensus of MAS ← uniformly jointly strongly connected &

$$\sum_{k=0}^{\infty} \alpha_k^- = \infty \text{ \& } \sum_{k=0}^{\infty} \alpha_k^+ \theta_k < \infty$$

Result 2: X_i is bounded and $0<\theta<\pi/4$ →

$$\sup_{x(0)} \limsup_{k\to\infty} |x_i(k)|_{X_0} < \infty$$

Numerical example



5. Conclusions

- Multi-agent system: promising
- Coordination problems beyond multi-agent consensus: set coordination, distributed optimization, …
- Set coordination for more generalized models: higher-order systems, event-triggered control, or stochastic or constrained models, …

Thank you！

☺