Synchronization of Complex Networks and Consensus of Multi-Agent Systems

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Outline

Outline

1.1 Synchronization of complex networks

- Network model

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} l_{ij} H(x_j),\ i = 1, 2, \cdots, N. \tag{1}$$

Network node $\dot{x}_i = f(x_i)$, Inner-linking function $H(\cdot)$, the coupling strength c, Laplace matrix $L = (l_{ij})$.
Eigenvalues of Laplace matrix $L = (l_{ij})$ (symmetrical):

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N. \tag{2}$$

Synchronization:

$$x_1(t) \to x_2(t) \to \cdots \to x_N(t),\ \text{as}\ t \to \infty. \tag{3}$$

Diffusive coupling $\longrightarrow$

$$x_1(t) \to x_2(t) \to \cdots \to x_N(t) \to s(t), \quad \dot{s}(t) = f(s(t)).$$

synchronization region (Master stability function method)

- Linearized equation: $\dot{\xi}_i = Df(s(t))\xi_i + c\sum_{j=1}^{N} l_{ij} DH(s(t))$
- Master stability equation:

$$\dot{\omega} = [Df(s(t)) + \alpha DH(s(t))]\omega. \tag{4}$$

- synchronization region S: the region of α such that the largest Lyapunov exponent of (4) $L_{max} < 0$.
- A stability condition of the synchronous state $s(t)$:

$$c\lambda_k \in S,\ k = 2, 3, \cdots, N. \tag{5}$$

- If the synchronous state $s(t)$ is an equilibrium point, the master stability equation becomes:

$$\dot{\omega} = [F + \alpha H]\omega. \tag{6}$$

The synchronization region S becomes the stable region of $F + \alpha H$ with respect to parameter α.

- Agent systems (linear model):

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \quad i = 1, \cdots, N. \end{aligned} \tag{7}$$

- An observer-based dynamic protocol:

$$\begin{aligned} &\dot{v}_i = (A + BK)v_i + cV\left(\sum_{j=1}^{N} l_{ij}C(v_i - v_j) - \sum_{j=1}^{N} l_{ij}(y_i - y_j)\right), \\ &u_i = Kv_i, \quad i = 1, \cdots, N, \end{aligned} \tag{8}$$

where $c > 0$ is the coupling strength, $L = (l_{ij})$ is the topological matrix, V and K are feedback gain matrices.

Multi-agent network

- Connecting (7) and (8) gives

$$\dot{\xi}_i = F\xi_i + c\sum_{j=1}^{N} l_{ij} H \xi_j, \quad i = 1, \cdots, N. \tag{9}$$

where $\xi_i = \begin{bmatrix} x_i \\ v_i \end{bmatrix}$, $\quad F = \begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix}$, $\quad H = \begin{bmatrix} 0 & 0 \\ -VC & VC \end{bmatrix}$.

- Node $\dot{\xi}_i = F\xi_i$, inner-linking matrix H.

Consensus

Given agent systems (7)， it is called that the protocol (8) solves the consensus, if the multi-agent network (9) satisfies

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \lim_{t\to\infty} v_i = 0, \quad \forall\, i, j = 1, 2, \cdots, N. \tag{10}$$

Consensus condition

Theorem 1

Given agent systems (7). Suppose that the communication topology L has a spanning tree, the protocol (8) solves the consensus problem if and only if $A + BK$ and $A + c\lambda_i VC$, $i = 2, \cdots, N$ are Hurwitz stable， where $\lambda_i, i = 2, \cdots, N$ are the non-zero eigenvalues of the Laplace matrix L.

- **Remark 1**: The protocol (8) can be viewed as a generalization of the observer-based controller for the traditional control systems. The separation principle still holds. By introducing a new parameter c, one can introduce a new concept "Consensus region" similar to the synchronization region in the complex networks.

Consensus region

- By Theorem 1, the stability of $A + c\lambda_i VC$ is very important for consensus. Viewing $c\lambda_i$ as a single parameter α leads to

Definition (Consensus region)

The region of α such that $A + \alpha VC$ is stable is called the consensus region.

- By Theorem 1, the condition for nework (9) achieving consensus is that $A + BK$ is stable and

$$c\lambda_k \in S, \quad k = 2, 3, \cdots, N.$$

- For undirected topology, the consensus region is on the real axis; for directed topology, the consensus region is in the complex plane. The region can be bounded, unbounded, or a set of several bounded regions, etc.

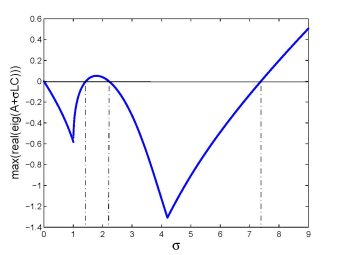
An example

- Given matrices

$$A = \begin{bmatrix} -1.4305 & 12.5142 & 3.3759 \\ 1 & -1 & 1 \\ -0.3911 & -5.5845 & -2.3369 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.8 & 9.6 & 2.6 \\ -0.3 & -5 & -1 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.8 & 9.6 \\ 0 & 0 \\ 0.3 & 5 \end{bmatrix}, K = \begin{bmatrix} -0.1720 & 13.3442 & 3.5464 \\ 0.9102 & -1.0954 & 0.7396 \end{bmatrix}.$$

- For undirected topology: the consensus region $S = (0, 1.4298) \cup (2.2139, 7.386)$.



Remarks

- From the above discussion, one can see that the problems in synchronization of complex networks and consensus of multi-agent systems can be studied in a unified framework. The concept "synchronization region" or "consensus region" is very important. The larger the consensus region, the easier the consensus. The consensus region shows the robustness of consensus.
- Actually, the topics in "synchronization of complex networks" is very popular in the community of physics. The concept of synchronization appeared very early. The physicists pay much attention on synchronization phenomena, and factors of influencing synchronization. The topics in "consensus of multi-agent systems" are popular in the area of dynamics and control. The control scientists pay much attention on the design of protocols to achieve consensus. These two concepts are closely related to each other.

Remarks

- In the field of complex network synchronization, the inner connecting matrix H is sometimes taken as $H = I$ by some authors. However, in the field of multi-agent consensus, the inner connecting matrix generally contains some designing variables, and generally it can not be taken as an identity matrix, e.g., in the above mentioned observer-type protocol, the inner connecting matrix is $H = \begin{bmatrix} 0 & 0 \\ -VC & VC \end{bmatrix}$.
- Viewing the multi-agent system connected by some different protocols, at this time, the inner connecting matrix will vary with the protocols and the single agent model. For example, with the higher order integer agent, the inner connecting matrix will have some special characteristics.

Outline

- The existence of n disconnected stable regions for $F + \alpha H$.

Theorem 2

For any natural number n, there are matrices F and H of order n such that $F + \alpha H$ has at least $[n/2] + 1$ disconnected stable regions with respect to parameter α.

- Main idea:
  - If a real polynomial is stable, then all its coefficients are positive.
  - Construct F and H such that the constant term of characteristic polynomial of $F + \alpha H$ is a polynomial with variable α of order n， the other coefficients are constants.

Constructing F and H

$$H = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 \\ -1 & 0 & \cdots & 0 \end{pmatrix}, \; F = \begin{pmatrix} -\beta & \beta_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \beta_{n-1} \\ -\beta_n & 0 & \cdots & -\beta \end{pmatrix},$$

$$\begin{aligned} \det(sI - F - \alpha H) &= (s+\beta)^n + (\alpha+\beta_1)(\alpha+\beta_2)\cdots(\alpha+\beta_n) \\ &= (s+\beta)^n - \beta^n + \beta^n + (\alpha+\beta_1)(\alpha+\beta_2)\cdots(\alpha+\beta_n). \end{aligned}$$

The case for given node dynamics (given F)

Theorem 3

For any given real stable matrix F of order n, suppose $\det(sI-F)=s^n+\gamma_{n-1}s^{n-1}+\cdots+\gamma_1 s+\gamma_0$, and every eigenvalue of F corresponds to only one Jordan form. If there is a scalar $\beta_0\neq 0$ such that $p(s)=s^n+\gamma_{n-1}s^{n-1}+\cdots+\gamma_1 s+\gamma_0-\beta_0$ is stable and $p(s)$ has n_i pairs of conjugate complex eigenvalues, then there exists a real matrix H such that $F+\alpha H$ has at least $[\frac{n-n_i}{2}]+1$ disconnected stable regions with respect to parameter α.

- Main idea: Using the Jordan form of F.

$$H_0=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix},\quad F_0=\begin{pmatrix} \xi_1 & 1 & 0 & 0 & \cdots & 0 \\ -\eta_1^2 & \xi_1 & \beta_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{03} & \beta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \beta_{n-1} \\ -\beta_n & 0 & 0 & 0 & \cdots & \lambda_{0n} \end{pmatrix}.$$

- Disconnected synchronization regions imply the possibility of intermittent synchronization behavior.

2.2 Convexity analysis

- Stability of $F+\alpha_1 H$ and $F+\alpha_2 H \Longrightarrow$ stability of $F+(\lambda\alpha_1+(1-\lambda)\alpha_2)H$, for all $0\leq\lambda\leq 1$?

Theorem 4

Suppose that $F+\alpha_1 H$ and $F+\alpha_2 H$ are stable, and the rank of H is 1. Let $H=bc$, where b is a column vector and c is a row vector with compatible dimensions, and (F,b) be controllable. Then, the following conditions are equivalent to each other:

- (i) $\lambda(F+\alpha_1 H)^{-1}+(1-\lambda)(F+\alpha_2 H)$ is stable for all $0\leq\lambda\leq 1$.
- (ii) There is a common matrix $P=P^T$ such that $P(F+\alpha_i H)+(F+\alpha_i H)^T P<0$, $i=1,2$.
- (iii) $(F+\alpha_1 H)(F+\alpha_2 H)$ does not have negative real eigenvalues.
- (iv) $1-\mathbf{Re}\{(\alpha_2-\alpha_1)c(jwI-F-\alpha_1 H)^{-1}b\}>0$, $\forall\, w\in\mathbf{R}$.

Further, if any one of (i)-(iv) holds, one has
($\star$) $F+(\lambda\alpha_1+(1-\lambda)\alpha_2)H$ is stable for all $0\leq\lambda\leq 1$.

Complex synchronization region

- Convexity analysis.

Theorem 5

Suppose that $F+\sigma_1 H$ and $F+\sigma_2 H$ are stable, and the rank of H is 1. Let $H=bc$, where b is a column vector and c is a row vector with compatible dimensions, and (F,b) be controllable. Then, the following statements are equivalent to each other:

- (i) $F+\frac{\sigma_1+\sigma_2}{2}H+\epsilon H$ is stable for all $\epsilon\in\mathbb{C}$, $|\epsilon|\le|\frac{\sigma_2-\sigma_1}{2}|$.
- (ii) $\|c(sI-F-\frac{\sigma_1+\sigma_2}{2}H)^{-1}b\|_\infty<\frac{2}{|\sigma_2-\sigma_1|}$.
- (iii) There is a common matrix $P=P_1+iP_2>0$ such that

$$P\left(F+\frac{\sigma_1+\sigma_2}{2}H+\epsilon H\right)+\left(F^T+\frac{\overline{\sigma}_1+\overline{\sigma}_2}{2}H^T+\overline{\epsilon}H^T\right)P^H<0,$$

$\forall\,\epsilon\in\mathbb{C}$, $|\epsilon|\le\left|\frac{\sigma_2-\sigma_1}{2}\right|$, where $\overline{\alpha}$ denotes the complex conjugate of α.

2.3 Unbounded synchronization region

- Real unbounded region

Theorem 6

Given a matrix $F \in \mathbf{R}^{n\times n}$, there exists a matrix $H \in \mathbf{R}^{n\times n}$ of rank 1 such that the stability region of $F + \alpha H$ with respect to parameter α contains $(-\infty, \alpha_1]$, $\alpha_1 < 0$, if and only if every unstable eigenvalue of F is corresponding to only one Jordan block.

- Complex unbounded region

Theorem 7

Given a matrix $F \in \mathbf{R}^{n\times n}$. Suppose that each unstable eigenvalue of F is corresponding to only one Jordan block. Then, there exists a matrix $H \in \mathbf{R}^{n\times n}$ of rank 1 such that $F + (x + yi)H$ is stable for all $x \in (-\infty, -1]$, $y \in (-\infty, +\infty)$.

2.4 Example: A Chua circuit network

- Chua's circuit node

$$\begin{array}{rcl} \dot{x}_{i1} & = & -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha(a x_{i1}^3 + b x_{i1}), \\ \dot{x}_{i2} & = & k x_{i1} - k x_{i2} + k x_{i3}, \\ \dot{x}_{i3} & = & -k\beta x_{i2} - k\gamma x_{i3}. \end{array} \quad (11)$$

Linearizing (11) at its zero equilibrium gives

$$\dot{x}_i = F x_i, \quad F = \begin{pmatrix} -k\alpha - k\alpha b & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{pmatrix}. \quad (12)$$

$k = 1, \alpha = -0.1, \beta = -1, \gamma = 1, a = 1, b = -25$. F is stable. Take the inner linking matrix

$$H = \begin{pmatrix} 0.8348 & 9.6619 & 2.6591 \\ 0.1002 & 0.0694 & 0.1005 \\ -0.3254 & -8.5837 & -0.9042 \end{pmatrix}.$$

Two disconnected stable regions: $S_1 = [-0.0099, 0]$ and $S_2 = [-2.225, -1)$.

Example: A Chua circuit network

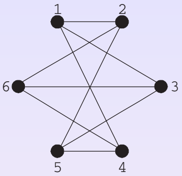


Fig. 2 Graph G_1

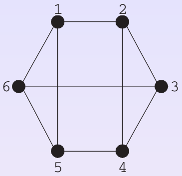


Fig. 3 Graph G_2

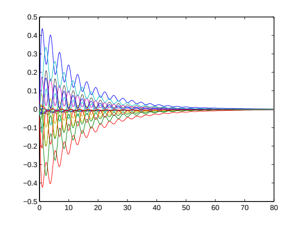


Fig. 4 Network on graph G_1.

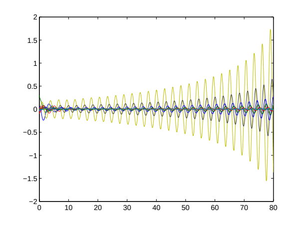


Fig. 5 Network on graph G_2.

- Combining graph theory and synchronization region together can discuss synchronization problems more completely.

2.4 Example: Satellite formation

- The linearized model of the i-th satellite with respect to the virtual satellite is given by Hill equations:

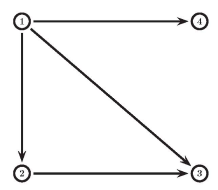
$$\begin{aligned} \ddot{x}_i - 2\omega_0 \dot{y}_i &= u_{x_i}, \\ \ddot{y}_i + 2\omega_0 \dot{x}_i - 3\omega_0^2 y_i &= u_{y_i}, \\ \ddot{z}_i + \omega_0^2 z_i &= u_{z_i}, \end{aligned}$$

- The control input to satellite i is designed as

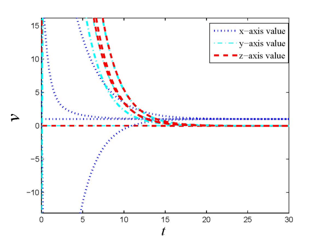
$$u_i = -A_1 h_i + c \sum_{j=1}^{N} a_{ij} \left(F_1 (r_i - h_i - r_j + h_j) + F_2 (\dot{r}_i - \dot{r}_j) \right),$$

where $A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\omega_0^2 & 0 \\ 0 & 0 & -\omega_0^2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 2\omega_0 & 0 \\ -2\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

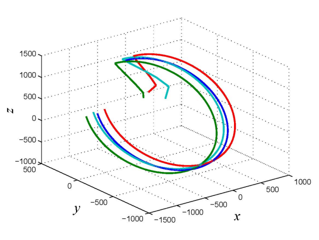
Suppose that the height of the virtual satellite is $a = 7359.5km$, and satellite 1 is the leader satellite. Take the initial state as $x_0 = 0$, $y_0 = 500$, $z_0 = 866$, $\dot{x}_0 = 1$, $\dot{y}_0 = 0$, $\dot{z}_0 = 0$. The four satellites maintain a square shape with a separation of 500 m in a plane tangent to the orbit of the virtual satellite by an angle 60 degree. Let $h_1 = (100, 100, 0)$, $h_2 = (-100, 100, 0)$, $h_3 = (100, 0, 173.21)$, $h_4 = (-100, 10, 173.21)$.



(a)



(b)



(c)

Remarks

- Generally in the field of complex network synchronization, the graph eigenvalue ratio λ_2/λ_N represents the synchronizability. Similarly, in the multi-agent systems, the consensusability problem can be studied. However, can the eigenration λ_2/λ_N really represent the synchronizability? When the synchronization region is disconnected, if every Laplace eigenvalue falls into a single part of synchronization region, the eigenratio represents nothing.
- The synchronizability problem should be studied by connecting the eigenratio and the synchronization region together. Especially for the directed networks, this problem can be very complicated.

Outline

3.1 Problem formulation

- Multi-agent systems with disturbances:

$$\dot{x}_i = Ax_i + Bu_i + D\omega_i, \quad i = 1, 2, \cdots, N, \tag{13}$$

- Distributed consensus protocol:

$$u_i = cK \sum_{j=1}^{N} a_{ij}(x_i - x_j), \tag{14}$$

- Let $x = [x_1^T, \cdots, x_N^T]^T$, $\omega = [\omega_1^T, \cdots, \omega_N^T]^T$. A multi-agent network is described by

$$\begin{aligned} \dot{x} &= (I_N \otimes A + cL \otimes BK)x + (I_N \otimes D)\omega, \\ z &= ((I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T) \otimes C)x, \end{aligned} \tag{15}$$

Let $T_{\omega z}$ denote the transfer function from ω to z.

Definition (H_∞ consensus)

For given scalar $\gamma > 0$， the protocol (14) solves the sub-optimal H_∞ consensus， if

- for $w_i = 0$, network (15) achieves consensus: $\lim_{t\to\infty} \|x_i - x_j\| = 0$;
- $\|T_{\omega z}\|_\infty < \gamma$.

Theorem 8

Suppose that the matrix L corresponds an undirected and connected graph. For given $\gamma > 0$， there exists a protocol (14) solving the sub-optimal H_∞ consensus problem if, and only if the following $N-1$ systems are stable and their H_∞ norm are smaller than γ:

$$\begin{aligned} \dot{\hat{x}}_i &= (A + c\lambda_i BK)\hat{x}_i + D\hat{\omega}_i, \\ \hat{z}_i &= C\hat{x}_i, \quad i = 2, 3, \cdots, N, \end{aligned} \tag{16}$$

where $\lambda_i, i = 2, \cdots, N$ are nonzero eigenvalues of Laplace matrix L.

3.3 H_∞ consensus region

- A new system:

$$\begin{aligned} \dot{\zeta} &= (A + \sigma BK)\zeta + D\omega_i, \\ z_i &= C\zeta, \end{aligned} \tag{17}$$

Definition (H_∞ consensus region)

The region S_γ of σ such that system (17) is stable and $\|\widehat{T}_{\omega_i z_i}\|_\infty < \gamma$ is called the H_∞ consensus region with index γ.

- For given $\gamma > 0$, the protocol (14) solves the H_∞ consensus problem if, and only $c\lambda_i \in S_\gamma$, $i = 2, 3, \cdots, N$.
- The region S_γ can be bounded, unbounded or disconnected.
- H_2 performance problems can be similarly studied

3.4 Unbounded H_∞ consensus region

Theorem 9

For given $\gamma > 0$, there exists a protocol (14) the H_∞ consensus region includes $\mathcal{S}_\gamma \triangleq [\tau,\ \infty)$, $\tau > 0$, if and only if there exists $P > 0$ such that

$$\begin{bmatrix} AP + PA^T - \tau BB^T & D & PC^T \\ D^T & -\gamma^2 I & 0 \\ CP & 0 & -I \end{bmatrix} < 0. \tag{18}$$

3.4 Unbounded H_∞ consensus region

Theorem 9

For given $\gamma > 0$, there exists a protocol (14) the H_∞ consensus region includes $\mathcal{S}_\gamma \triangleq [\tau, \infty)$, $\tau > 0$, if and only if there exists $P > 0$ such that

$$\begin{bmatrix} AP + PA^T - \tau BB^T & D & PC^T \\ D^T & -\gamma^2 I & 0 \\ CP & 0 & -I \end{bmatrix} < 0. \tag{18}$$

Algorithm

Constructing the protocol for solving the H_∞ consensus problrms;

1) Solve the LMI (18) to get $P > 0$ and $\tau > 0$;
2) Choose $K = -\frac{1}{2}B^T P^{-1}$;
3) Choose the coupling strength $c \geq \frac{\tau}{\min\limits_{i=2,\cdots,N} \lambda_i}$, where λ_i, $i = 2, 3, \cdots, N$ are the nonzero eigenvalues of the Laplace matrix L.

3.5 Complexity of H_2 performance changes

- Node dynamics:

$$\dot{x}_i = Fx_i + B_1 u, \quad y = C_1 x_i. \tag{19}$$

- Complex network:

$$\begin{aligned} \dot{x}_i &= Fx_i - c\sum_{j=1}^{N} l_{ij} H x_j + B_1 u, \\ y &= \sum_{j=1}^{N} C_1 x_j. \\ i &= 1, 2, \cdots, N. \end{aligned} \tag{20}$$

A special network topology

- A special matrix L:

  $L_{N\times N} = (l_{ij})_{N\times N}$, $l_{ii} = 0$; $l_{ij} = -1$, if $i > j$, $j \leq i-2$ and j is odd; otherwise $l_{ij} = 1$.

  (21)

- Example:

When $N = 5$, $$L_{5\times 5} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

When $N = 6$, $$L_{6\times 6} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0 & 1 & 1 \\ -1 & 1 & -1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

Characteristics of this new matrix L

Lemma When N is even, different eigenvalues of $L_{N\times N}$ are -1 and 1, and $L_{N\times N}$ is similar to

$$\Lambda_N = \begin{pmatrix} -I_{\frac{N}{2}} & 0 \\ 0 & J_{\frac{N}{2}} \end{pmatrix},$$

where $I_{\frac{N}{2}}$ is a unit matrix of order $\frac{N}{2}\times\frac{N}{2}$; $J_{\frac{N}{2}}$ is Jordan block of order $\frac{N}{2}\times\frac{N}{2}$:

$$J_{\frac{N}{2}} = \begin{pmatrix} 1 & 2 & & \\ & \ddots & \ddots & \\ & & \ddots & 2 \\ & & & 1 \end{pmatrix}.$$

When N is odd, different eigenvalues of $L_{N\times N}$ are -1, 1 and 0, and L is similar to $\mathrm{diag}(\Lambda_{N-1}, 0)$.

Accumulation of H_2 norm

- Node equation:

$$\dot{x}_1 = Fx_1 + B_1 u_1, \qquad y_1 = C_1 x_1, \tag{22}$$

$$F = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ \frac{1}{0.866} \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

- A special network:

$$\dot{x} = Ax + Bw, \qquad y = Cx, \tag{23}$$

where
$A = (I_N \bigotimes F + L \bigotimes H)$, $H = B_1 C_1$, $B = \begin{pmatrix} B_1^T & \cdots & B_1^T \end{pmatrix}^T$,
and $C = \begin{pmatrix} C_1 & \cdots & C_1 \end{pmatrix}$,

- A is stable if A_1, $A_1 - H$ and $A_1 + H$ are stable.

Accumulation of H_2 norm

- (1) When $N = 1$, H_2 norm of system (23) is 1.
- (2) When $N = 2$, H_2 norm of system (23) is **4.4740**.
- (3) When $N = 3$, H_2 norm of system (23) is **8.2455**.
- (4) When $N = 4$, H_2 norm of system (23) is **31.1467**.

- (6) When $N = 6$, H_2 norm of system (23) is **230.4775**.
- (8) When $N = 8$, H_2 norm of system (23) is **1.7971e+003**.
- (10) When $N = 10$, H_2 norm of system (23) is **1.4385e+004**.
- (12) When $N = 12$, H_2 norm of system (23) is **1.1687e+005**.
- (14) When $N = 14$, H_2 norm of system (23) is **9.5860e+005**.
- (16) When $N = 16$, H_2 norm of system (23) is **7.9151e+006**.
- (20) When $N = 20$, H_2 norm of system (23) is **5.4700e+008**, which is larger than 2^{20}.

- Diffusive coupled complex network:

$$l_{ii} = -\sum_{j=1, j\neq i}^{N} l_{ij}.$$

Network (23) with N nodes, its H_2 norm is N (this can be proved strictly).

- Anti-symmetrical coupled complex network:

$$l_{ii} = 0,\ l_{ij} = -L_{ji}.$$

The H_2 norm of the corresponding network increases slowly.

- The changes of the H_2 norms of different complex network are very different.

3.6 The impact of H_2 norm on synchronization

- Consider Lur'e node system:

$$\begin{cases} \dot{x}_1 = (F - 2H)x_1 + B_{01} f_1(y_1), \quad x_1(0) = B_{01}, \\ y_1 = C_{01} x_1, \end{cases} \tag{24}$$

where

$$F = \begin{pmatrix} 0 & 1 \\ -4 & -2.5 \end{pmatrix}, \quad B_{01} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_{01} = \begin{pmatrix} 2 & 2 \end{pmatrix}, \quad H = B_{01}C$$

$f_1(y_1) = |y_1 + 1| - |y_1 - 1|$, f_1 satisfies the sector condition:

$$0 \leq \frac{f_1(y_1)}{y_1} \leq 2, \quad f_1(0) = 0. \tag{25}$$

The impact of H_2 norm on synchronization

- Consider a network with the regulated output:

$$\begin{cases} \dot{x} = (I_N \bigotimes (F - 2H) + L_L \bigotimes H)x + Bf(y), \quad x(0) = E_N \bigotimes x_1(0), \\ y = C_1 x, \\ z = C_2 x, \end{cases} \tag{26}$$

- Linearizing netowrk (26):

$$\begin{cases} \dot{x} = (I_N \bigotimes F + L_L \bigotimes H)x, \quad x(0) = E_N \bigotimes x_1(0), \\ z = C_2 x. \end{cases} \tag{27}$$

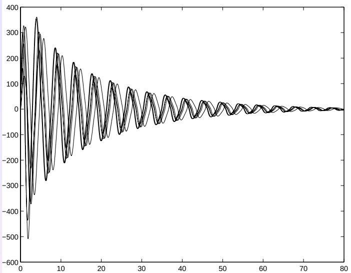
By LQR method, network (27) can be viewed as:

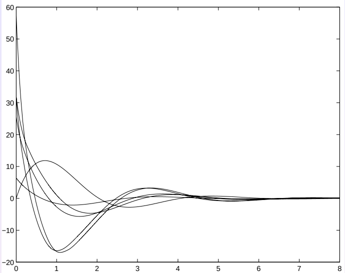
$$\begin{cases} \dot{x} = (I_N \bigotimes F + L_L \bigotimes H)x + E_N \bigotimes x_1(0)\delta(t), \quad x(0_-) = 0, \\ z = C_2 x, \end{cases} \tag{28}$$

The output energy of netowrk (27) is:

$$\|C_2(sI - I_N \bigotimes F - L_L \bigotimes H)^{-1} E_N \bigotimes x_1(0)\|_2.$$

Output synchronization





Network with norm accumulated topology. Network with diffusive topology.

- The larger the H_2 norm, the slower the synchronization.
- The problem related to H_∞ norm can be similarly studied.

Outline

Other related problems

- Globally synchronization region of nonlinear networks (global consensus region of nonlinear multi-agent networks);
- H_2 and H_∞ performance region);
- Directed networks $\rightarrow$ complex synchronization region;
- Other complex characteristics of complex networks;
- Combining graph theory with synchronization regions;
- Practical applications.

Related papers

1. Zhongkui Li, Zhisheng Duan, Guanrong Chen, On H-infinity and H2 Performance Regions of Multi-Agent Systems, Automatica, 47, 797-803, 2011
2. Zhongkui Li, Zhisheng Duan, Guanrong Chen, Global synchronized region of linearly coupled Lur'e systems, International Journal of Control, 84(2), 216-227, 2011
3. Zhongkui Li, Zhisheng Duan, Guanrong Chen, Lin Huang, Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint, IEEE Transactions on Circuits and Systems-I, 57(1), 213-224, 2010
4. Zhisheng Duan, Jinzhi Wang, Guanrong Chen, Lin Huang, H2 norm accumulation and its impact on synchronization of complex dynamicl networks, International Journal of Control, 82(12), 2356-2364, 2009
5. Chao Liu, Zhisheng Duan, Guanrong Chen, Lin Huang, L2 norm performance index of synchronization and LQR control synthesis of complex networks, Automatica, 45, 1879-1885, 2009
6. Zhisheng Duan, Guanrong Chen, Lin Huang, Disconnected synchronization regions of complex dynamical networks, IEEE Transactions on Automatic Control, 54(4), 845-849, 2009
7. Zhisheng Duan, Guanrong Chen, Lin Huang, Synchronization of weighted networks and complex synchronization regions, Physics letters A, 372，3741－3751, 2008
8. Zhisheng Duan, Chao Liu, Guanrong Chen，Network synchronizability analysis: The theory of subgraphs and complementary graphs, Physica D, 237(7）, 1006-1012, 2008
9. Zhisheng Duan, Jinzhi Wang，Guanrong Chen, Lin Huang, Stability analysis and decentralized control of a class of complex dynamical networks, Automatica,44, 1028-1035, 2008
10. Zhisheng Duan, Guanrong Chen，Lin Huang, Complex network synchronizability: Analysis

Many thanks!
Any questions?