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Distributed Consensus with Limited Communication Data Rate

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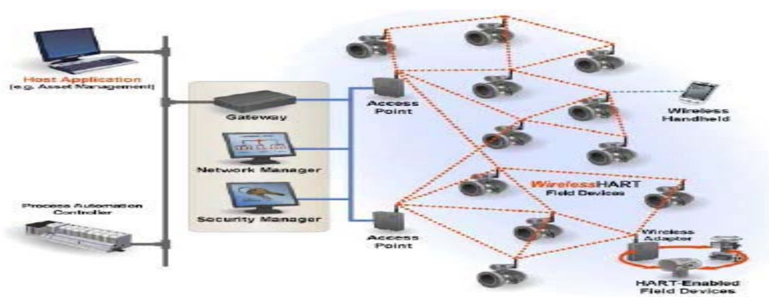
Outline

- Background and motivation
- Case with time-invariant topology
  - protocol design and closed-loop analysis
  - performance limit analysis
  - communication energy minimization
- Case with time-varying topologies
  - protocol design and closed-loop analysis
  - finite duration of link failures
- Concluding remarks

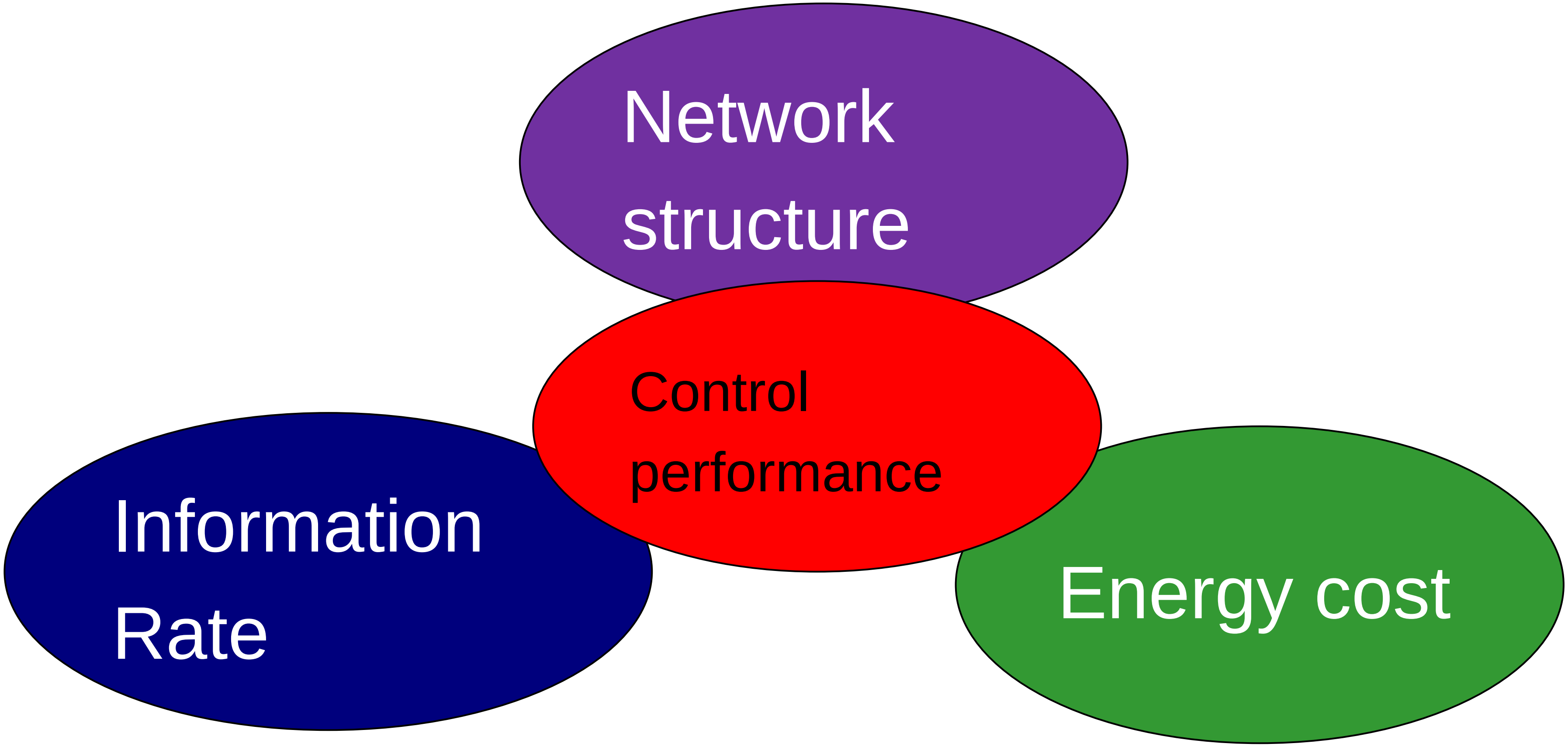
Background and Motivation

NCS and Multi-Agent Systems



L. Xie, CCC 2011

Communication and cooperation become essential factors of control systems



Communication and energy constraints are important factors to be investigated for cooperation over multi-agent networks

Literature: communication limited single-agent control

- Minimum data rate theorem
  - Nair & Evans, SCL, 2000
  - Tatikonda & Mitter, TAC, 2004
  - Nair & Evans, SICON, 2004
  - Nair et al. TAC, 2004
- Coarsest quantization
  - Elia & Mitter, TAC, 2001
  - Fu & Xie, TAC, 2001
  - Tsumura, Ishii & Hoshina, Atumatica, 2009

Literature: communication limited multi-agent coordination

- Noisy analog communication
  - ✓ Huang & Manton, SICON, 2009
  - ✓ Kar & Moura, TSP, 2009
  - ✓ Li & Zhang, Automatica, 2009
  - ✓ Huang et al. Automatica 2010
  - ✓ Li & Zhang, TAC, 2011
- Integer-valued consensus
  - ✓ Kashyap Basar & Srikant, Automatica, 2007
  - ✓ Nedic et al., TAC, 2009

Literature: communication limited multi-agent coordination

- Consensus with quantized communication
  - Frasca et al. IJNRC, 2009
  - Carli & Bullo, SICON, 2009
  - Carli et al. Automatica, 2010
  - Carli et al. IJNRC, 2010
- Consensus with quantized measurement
  - Dimarogonas & Johansson, Automatica 2010
  - Chen Lewis & Xie, Automatica 2011

- Problem 1:

How many bits does each pair of neighbors need to exchange at each time step to achieve consensus of the whole network ?

- Problem 2:

What is the relationship

between the control performance (i.e. convergence rate)

and the communication data rate?

- Problem 3:

What is the relationship of the communication energy cost, the convergence rate and the data rate ?

- Problem 4:

How to design encoders and decoders when the communication topology is time-varying or the transmission is unreliable (packet dropout, delay)?

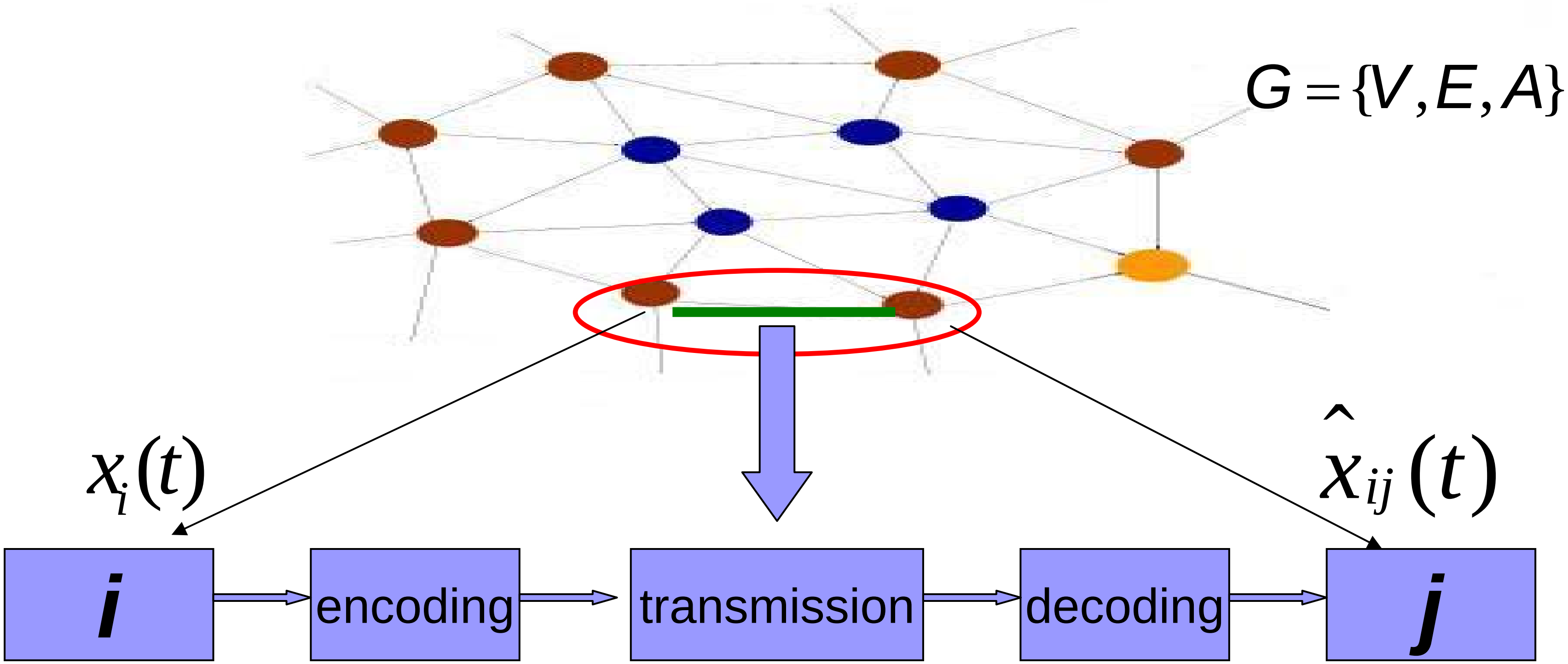
Collaborators

- Xie Lihua, School of EEE, Nanyang Technological University, Singapore.
- Fu Minyue, School of EE&CS, University of Newcastle, Australia.
- Zhang Ji-Feng, AMSS, Chinese Academy of Sciences, China
- Liu Shuai, School of EEE, Nanyang Technological University, Singapore.

T. Li, M. Fu, L. Xie, J. F. Zhang, IEEE TAC, February, 2011

Case with time-invariant topology

$$x_i(t+1) = x_i(t) + u_i(t),\ t = 0,1,\ldots,\ i = 1,2\ldots,N$$

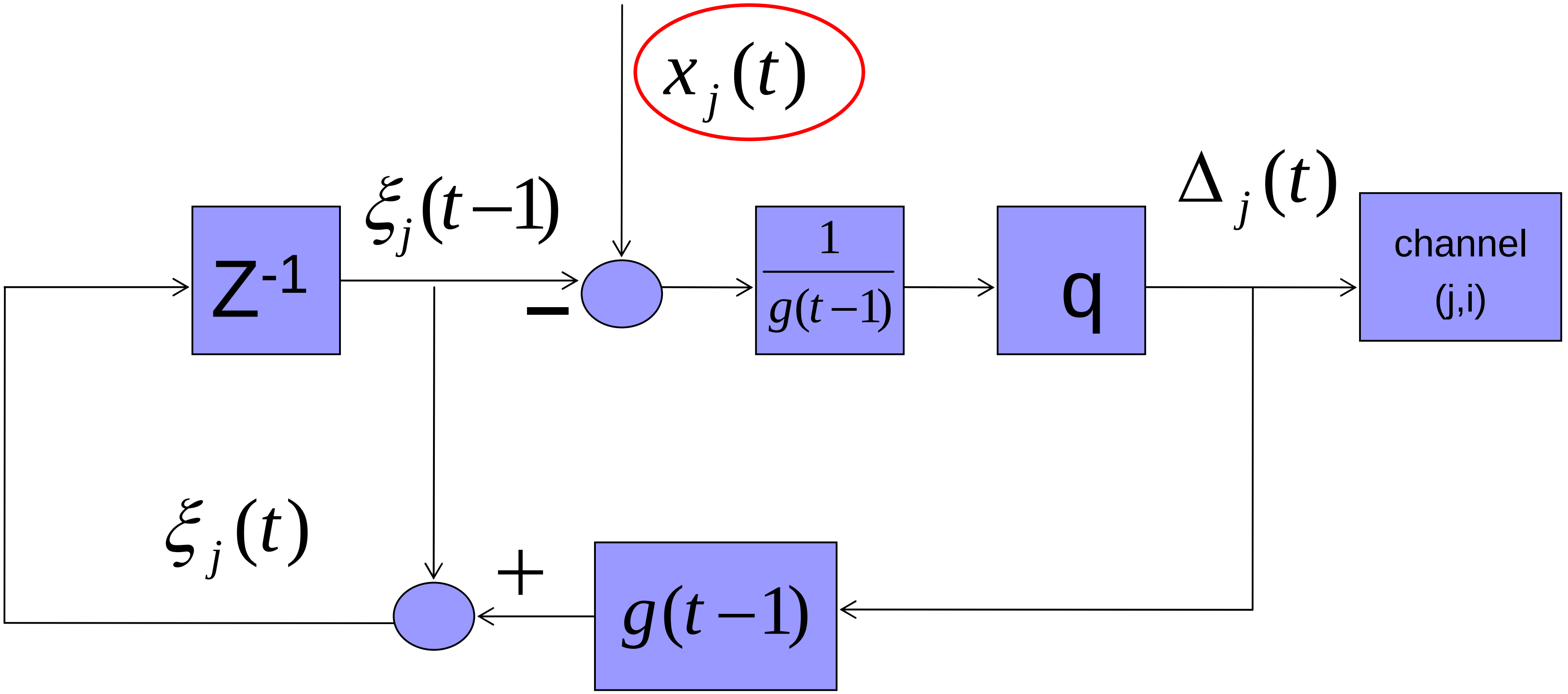


States are real-valued， but only finite bits of information are transmitted at each time-step

- Difficulties
  - quantization error
    - divergence of the states
  - finite-level quantizer
    - nonlinearity, unbounded quantization error
- Key points
  - design proper encoder, decoder and protocol
    - stabilize the whole network
    - eliminate the effect of quantization error on achieving exact consensus

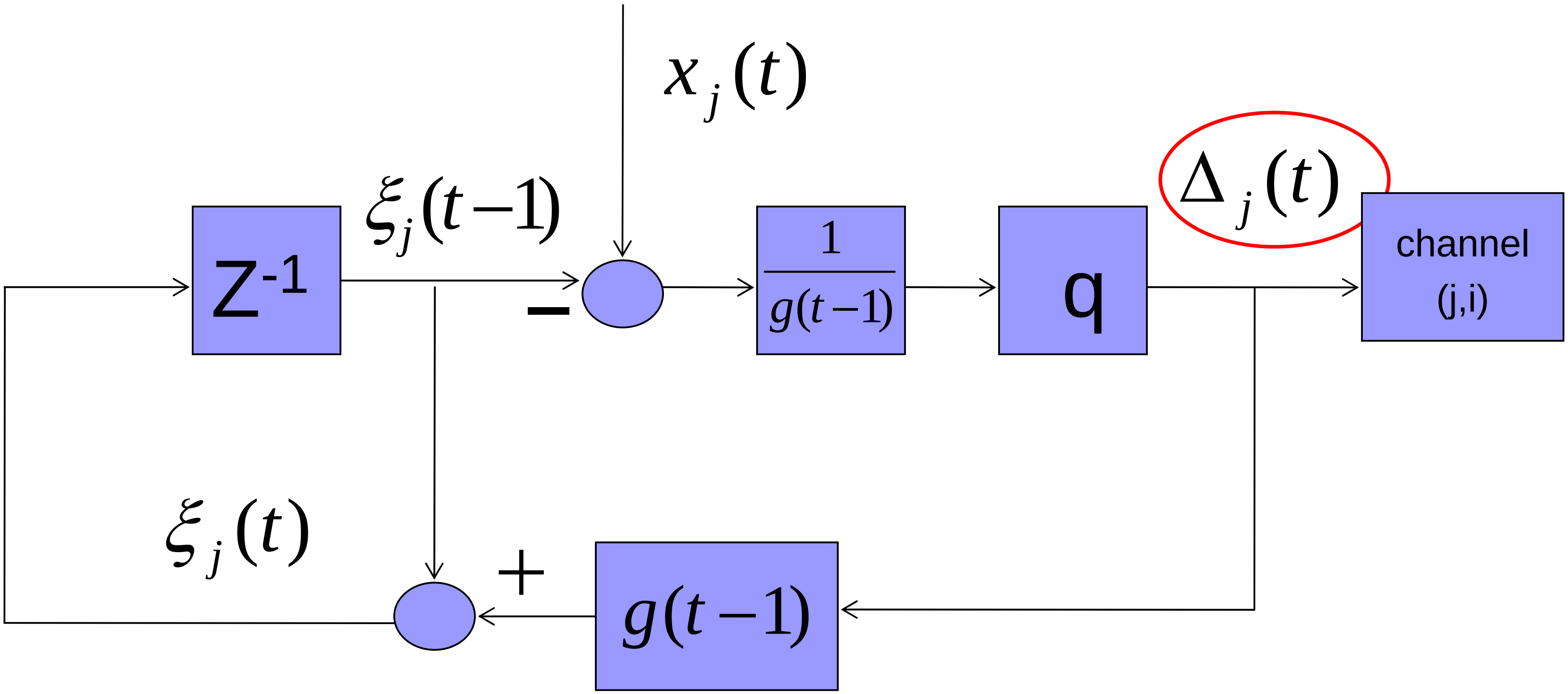
Distributed protocol

- encoder ϕ_j



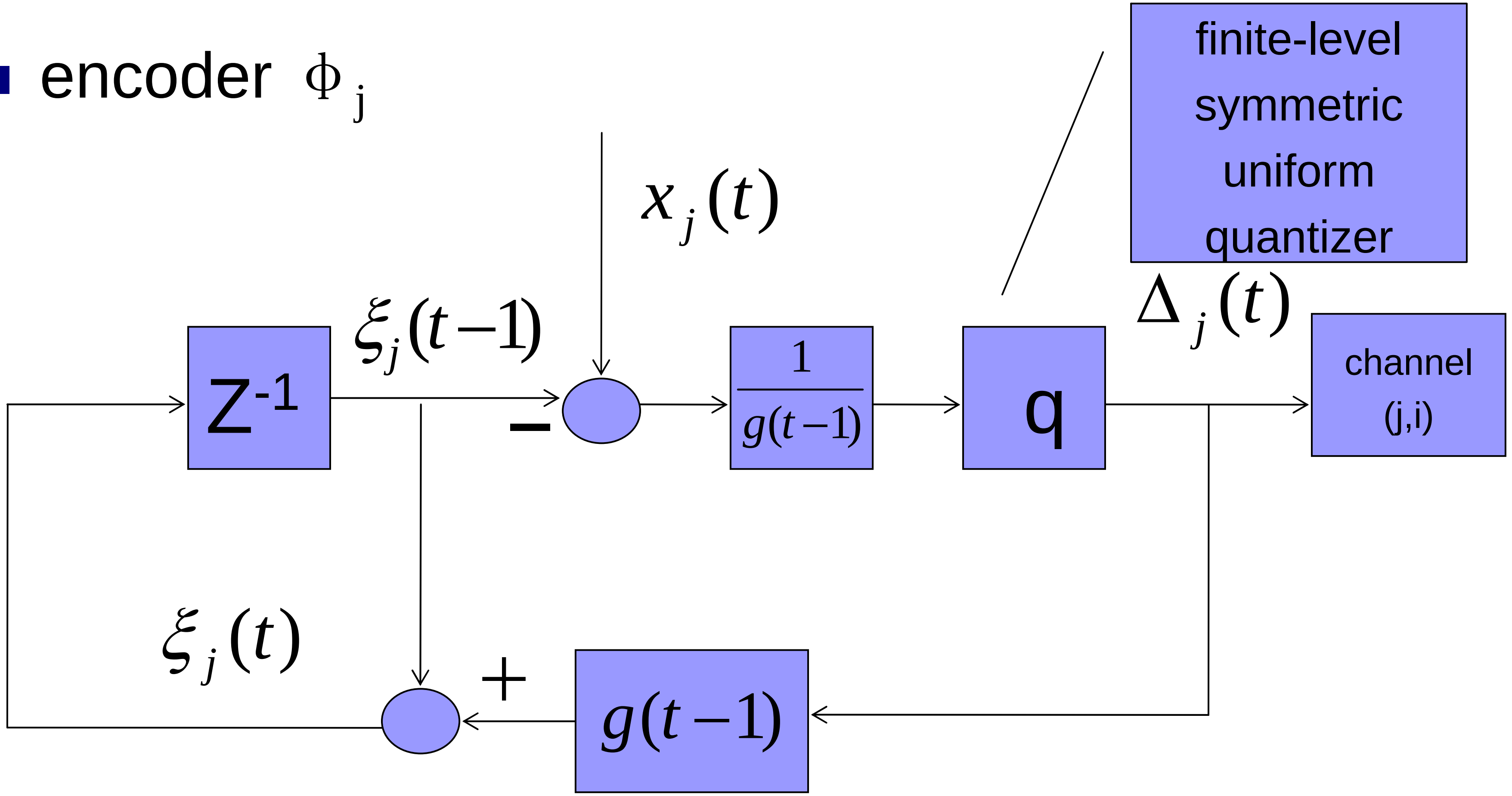
Distributed protocol

- encoder Φ_j



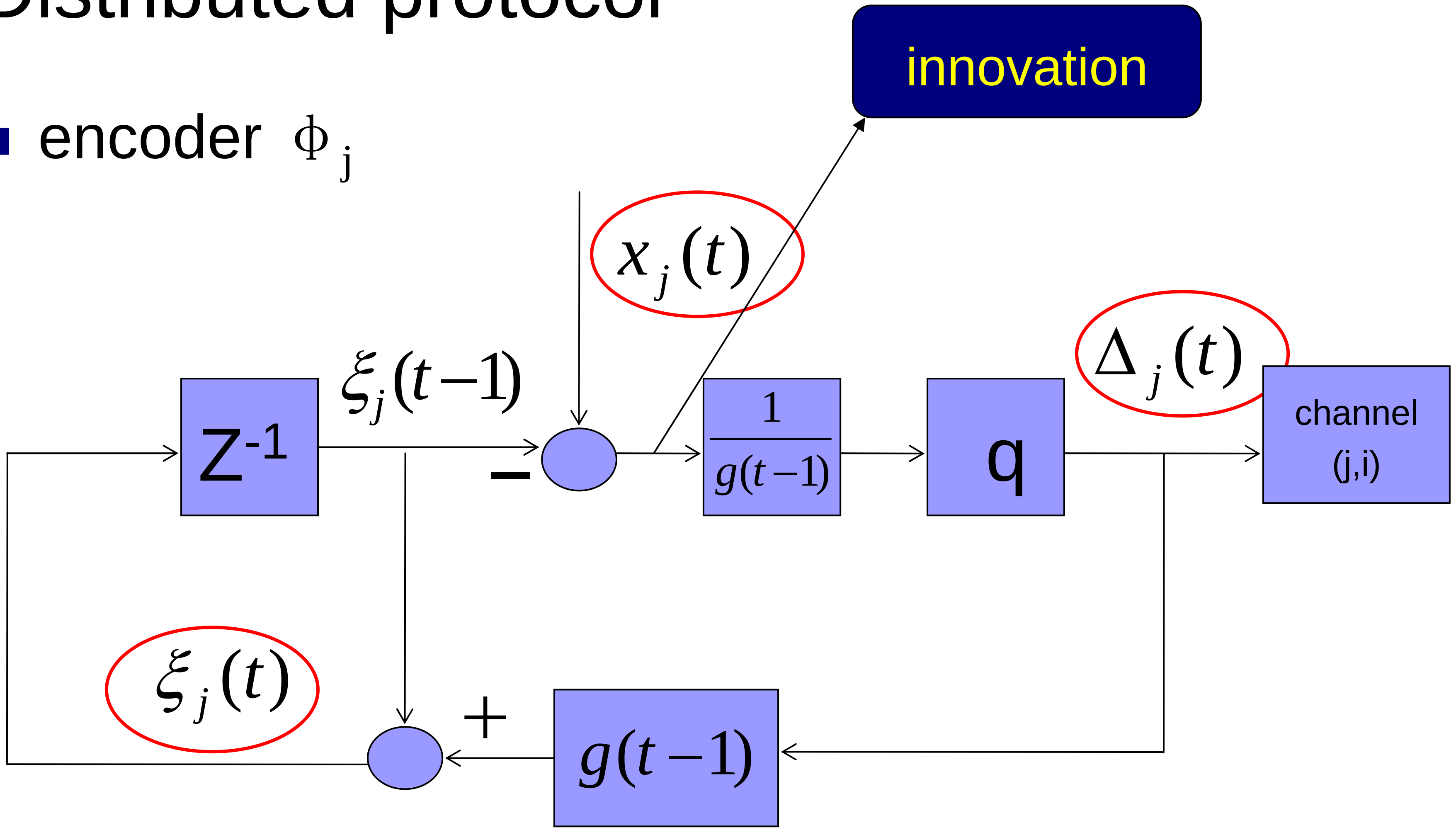
Distributed protocol

- encoder Φ_j



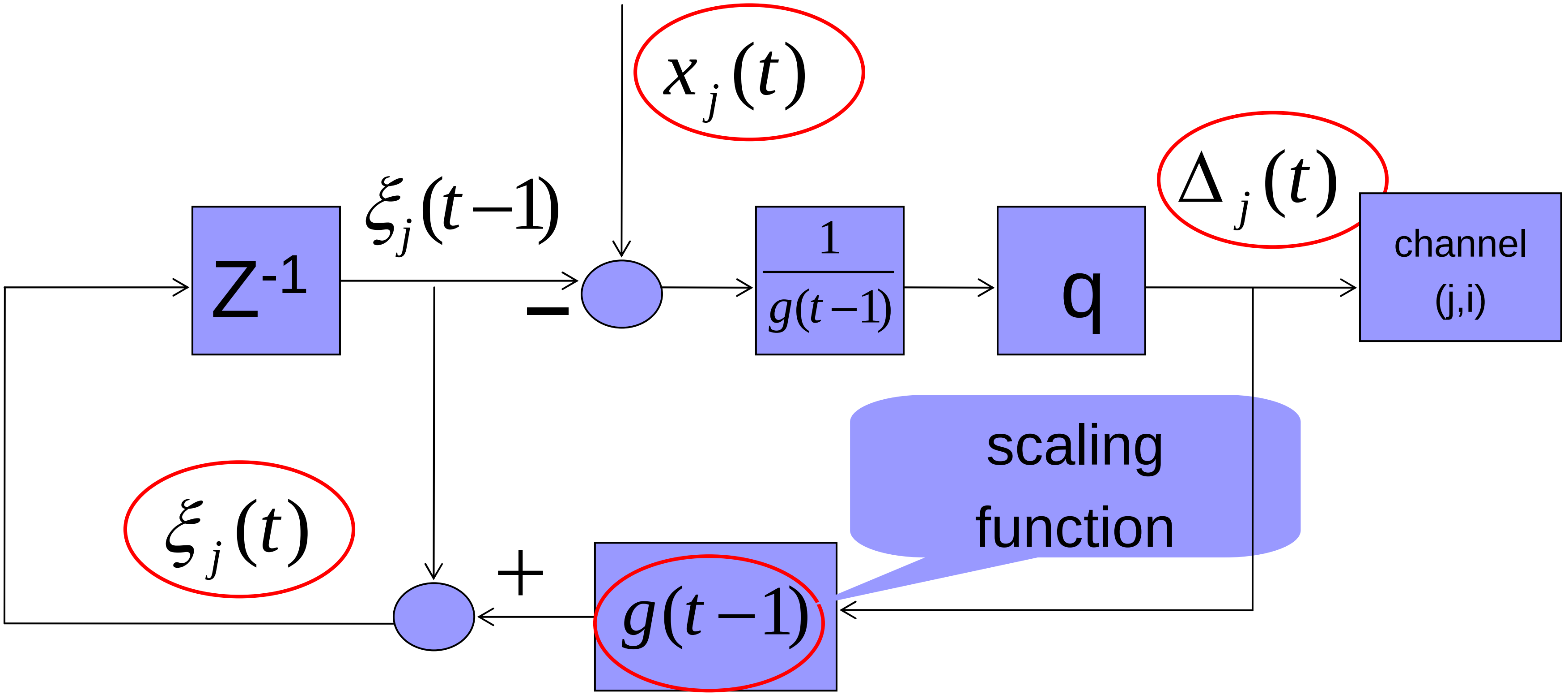
Distributed protocol

- encoder Φ_j



Distributed protocol

- encoder Φ_j



Distributed protocol

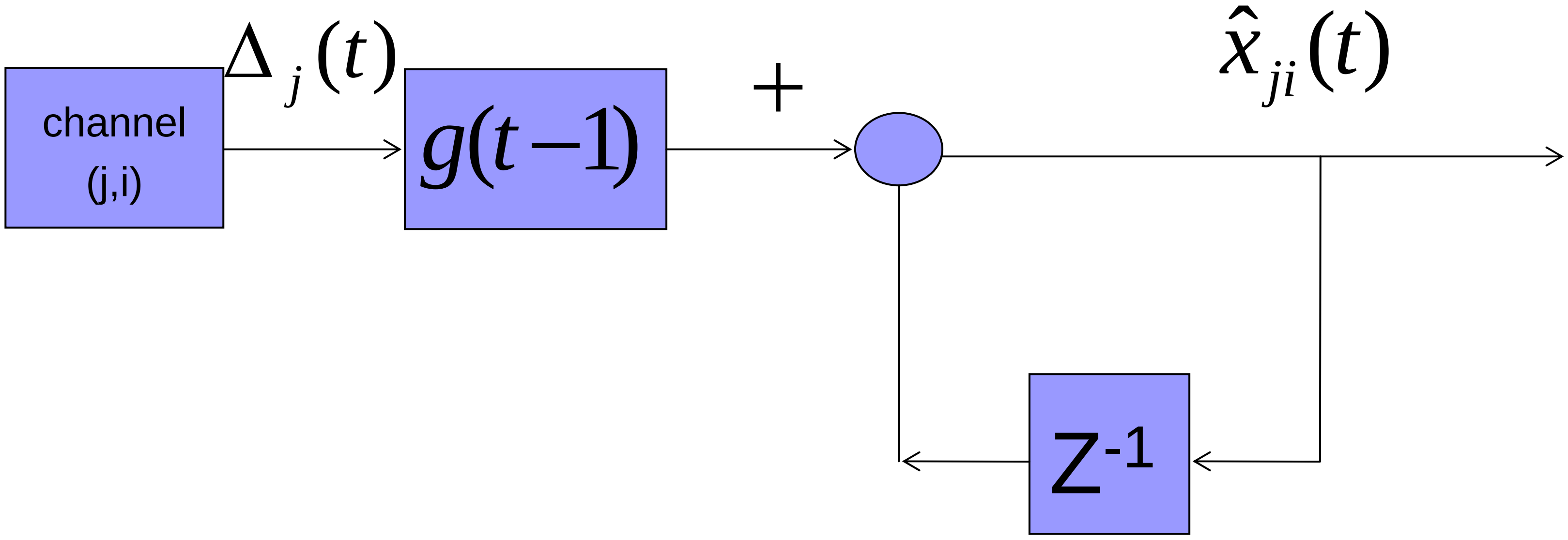
- Symmetric uniform quantizer

(2K+1) levels
$\lceil \log_2(2K) \rceil$ bits

$$q(y) = \begin{cases} 0, & -1/2 < y < 1/2 \\ i, & (2i-1)/2 \le y < (2i+1)/2 \\ K, & y \ge (2K-1)/2 \\ -(q(-y)), & y \le -1/2 \end{cases}$$

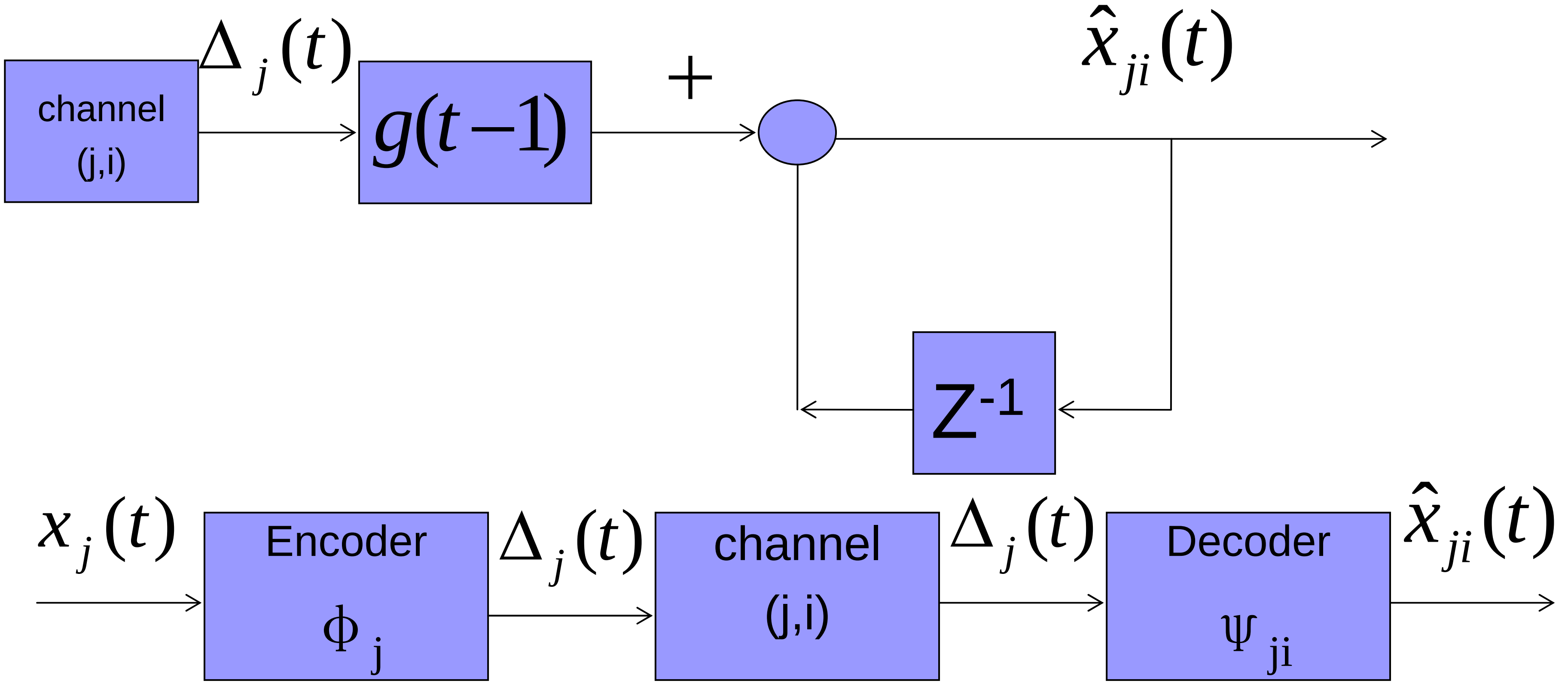
Distributed protocol

- decoder ψ_{ji}



Distributed protocol

- decoder ψ_{ji}



Distributed protocol

- protocol

$$u_i(t) = h \sum_{j \in N_i} a_{ij} (\hat{x}_{ji}(t) - \xi_i(t)), \ t = 0, 1, \ldots,$$

$$\sum_{i=1}^{N} u_i(t) \equiv 0$$

$$i = 1, 2, \ldots, N$$

$$u_i(t) = h \sum_{j \in N_i} a_{ij} (x_j(t) - x_i(t))$$
$$- h \sum_{j \in N_i} a_{ij} (x_j(t) - \hat{x}_{ji}(t))$$
$$+ h \sum_{j \in N_i} a_{ji} (x_i(t) - \hat{x}_{ij}(t))$$

average preserved

control gain: h

quantization levels: $2K+1$

scaling fucntion: $g(t)$

Main assumptions

A1) G is a connected graph

A2) $\max_{1 \le i \le N} |x_i(0)| \le C_x, \ \max_{1 \le i \le N} |\delta_i(0)| \le C_\delta$

Denote

$$\rho_h = \max_{2 \le i \le N} |1 - h\lambda_i|$$

$$\delta(t) = \left[x_1(t) - \frac{1}{N}\sum_{j=0}^{N} x_j(0), \cdots, x_N(t) - \frac{1}{N}\sum_{j=0}^{N} x_j(0) \right]^T$$

For a given control performance (i.e. convergence rate), what is the communication data rate required ?

Lemma. Suppose A1)-A2) hold. For any given

$h \in (0, 2/\lambda_N)$ and $\gamma \in (\rho_h, 1)$

If $2K+1 \geq 2K_1(h,\gamma)+1$

then $\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0),\ i=1,2,\ldots,N$

by taking $g(t) = g_0\gamma^t$ with

$$g_0 > \max\left\{\frac{C_x}{K+1/2},\ \frac{2(\gamma-\rho_h)(C_\delta\gamma + hC_x\lambda_N)}{h\lambda_N}\right\}$$

where

$$K_1(h,\gamma) = \left\lfloor \frac{\sqrt{N}h^2\lambda_N^2}{2\gamma(\gamma-\rho_h)} + \frac{1+2hd^*}{2\gamma} - \frac{1}{2} \right\rfloor + 1$$

convergence rate:

$$\|\delta(t)\| = O(\gamma^t),\ t \to \infty$$

Lemma. Suppose A1)-A2) hold. For any given

$h \in (0, 2/\lambda_N)$ and $\gamma \in (\rho_h, 1)$

$$\text{If } \; 2K+1 \geq 2K_1(h,\gamma)+1$$

$$\text{then} \quad \lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0), \; i = 1, 2, \ldots, N$$

by taking $g(t) = g_0\gamma^t$ with

$$g_0 > \max\left\{ \frac{C_x}{K+1/2}, \; \frac{2(\gamma-\rho_h)(C_\delta\gamma + hC_x\lambda_N)}{h\lambda_N} \right\}$$

Convergence rate for perfect communication

Lemma. Suppose A1)-A2) ho

$h \in (0, 2/\lambda_N)$ and $\gamma \in (\rho_h, 1)$

$$\text{If } \; 2K+1 \geq 2K_1(h,\gamma)+1$$

$$\text{then} \quad \lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0), \;\; i = 1, 2, \ldots, N$$

by taking $g(t) = g_0 \gamma^t$ with

$$g_0 > \max\left\{ \frac{C_x}{K+1/2}, \;\; \frac{2(\gamma-\rho_h)(C_\delta \gamma + hC_x\lambda_N)}{h\lambda_N} \right\}$$

$$\gamma \to \rho_h$$

$$K_1(h,\gamma) \to \infty$$

$$K_1(h,\gamma) = \left\lfloor \frac{\sqrt{N}\,h^2 \lambda_N^2}{2\gamma(\gamma - \rho_h)} + \frac{1 + 2hd^*}{2\gamma} - \frac{1}{2} \right\rfloor + 1$$

- To save communication bandwidth

⇓

✓ Minimize the number of quantization levels $2K+1$

$$\lim_{h\to 0^+}\lim_{\gamma\to 1^-}\frac{\sqrt{N}h^2\lambda_N^2}{2\gamma(\gamma-\rho_h)}+\frac{1+2hd^*}{2\gamma}=\frac{1}{2}$$

⇓

$$\lim_{h\to 0^+}\lim_{\gamma\to 1^-}\log_2\lceil 2K_1(h,\gamma)\rceil=1$$

Theorem. Suppose A1)-A2) hold. For any given $K \geq 1$

let

$$\Omega_K = \left\{ (h,\gamma) \mid 0 < h < \frac{2}{\lambda_2 + \lambda_N},\ \rho_h < \gamma < 1,\ M_1(h,\gamma) < K + \frac{1}{2} \right\}$$

Then (i) Ω_K is not empty

(ii) for any given $(h,\gamma) \in \Omega_K$, let $g(t) = g_0 \gamma^t$

⇓

$$\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0),\ i = 1, 2, \ldots, N$$

For a connected network, average-consensus can be achieved with exponential convergence rate base on a single-bit exchange between each pair of neighbors at each time step

For a given communication data rate, what is the best control performance we can achieve ?

Performance limit analysis

Theorem. Suppose G is connected, then

for any given $K \geq 1$, we have

$$\lim_{N \to \infty} \frac{1 - \inf_{(h,\gamma) \in \Omega_K} \gamma}{1 - \exp\left\{ -\frac{K Q_N^2}{2\sqrt{N}} \right\}} = 1$$

where

$$Q_N = \frac{\lambda_2}{\lambda_N}$$

Performance limit analysis

Theorem. Suppose G is connected, then

for any given $K \geq 1$, we have

$$\lim_{N \to \infty} \frac{1 - \inf_{(h,\gamma) \in \Omega_K} \gamma}{1 - \exp\left\{ -\frac{K Q_N^2}{2\sqrt{N}} \right\}} = 1$$

where

$$Q_N = \frac{\lambda_2}{\lambda_N}$$

Synchronizability

Barahona & Pecora
PRL 2002

Donetti et al.
PRL 2005

The highest asymptotic convergence rate increases as the number of quantization levels and the synchronizability increase and decreases as the network expands

$$\inf_{(h,\gamma)\in\Omega_K} \gamma \approx \exp\left\{-\frac{KQ_N^2}{2\sqrt{N}}\right\}$$

How to joint design communication data rate and control parameters to optimize energy cost ?

Power limitation is a critical constraint

Energy to transmit 1 bit $\approx$ Energy for 1000-3000 operations

Shnayder et al. 2004

How to select the number of quantization levels (data rate) and convergence rate to minimize the communication energy

Model of communication energy

- convergence time constant

$$\tau_{asym} = (\ln(1/r_{asym}))^{-1}$$

$$r_{asym} = \sup_{X(0) \neq JX(0)} \lim_{t \to \infty} \left(\frac{\| X(t) - JX(0) \|}{\| X(0) - JX(0) \|} \right)^{1/t}$$

Consensus error decreasing by 1/e

Xiao & Boyd 2004

Model of communication energy

Energy for transmitting a single bit

Energy for receiving a single bit

$$\Phi(K) = \lceil \log_2(2K) \rceil (c_1 |V| + 2c_2 |E|)\tau_{asym}$$

The faster, the larger

The faster, the smaller

Fast convergence does not imply power saving. The optimization of the total communication cost leads to tradeoff between number of quantization levels and convergence rate.

Suboptimal solution

Energy for transmitting a single bit

Energy for receiving a single bit

$$\Phi(K) = \lceil \log_2 (2K) \rceil (c_1 |V| + 2c_2 |E|)\tau_{asym}$$

$$K = K_1(h, \gamma), \ r_{asym} \le \gamma$$

Optimization w.r.t. γ

$$\Phi(K) \le (c_1 |V| + 2c_2 |E|) \lceil \log_2(2K_1(h,\gamma)) \rceil [\ln(1/\gamma)]^{-1}$$

$$B_h(\gamma)$$

T. Li, L. Xie, Automatica, September, 2011

Case with time-varying topologies

Channel uncertainties：

- ✓ Link failures
- ✓ Packet dropouts
- ✓ Channel noises

How to design **robust encoding-decoding** schemes and **control protocols** against channel uncertainties

Network model

$G(t) = \{V, A(t)\}, t = 1, 2...,$

$A(t) = [a_{ij}(t)]$ is the adjacency matrix of $G(t)$,

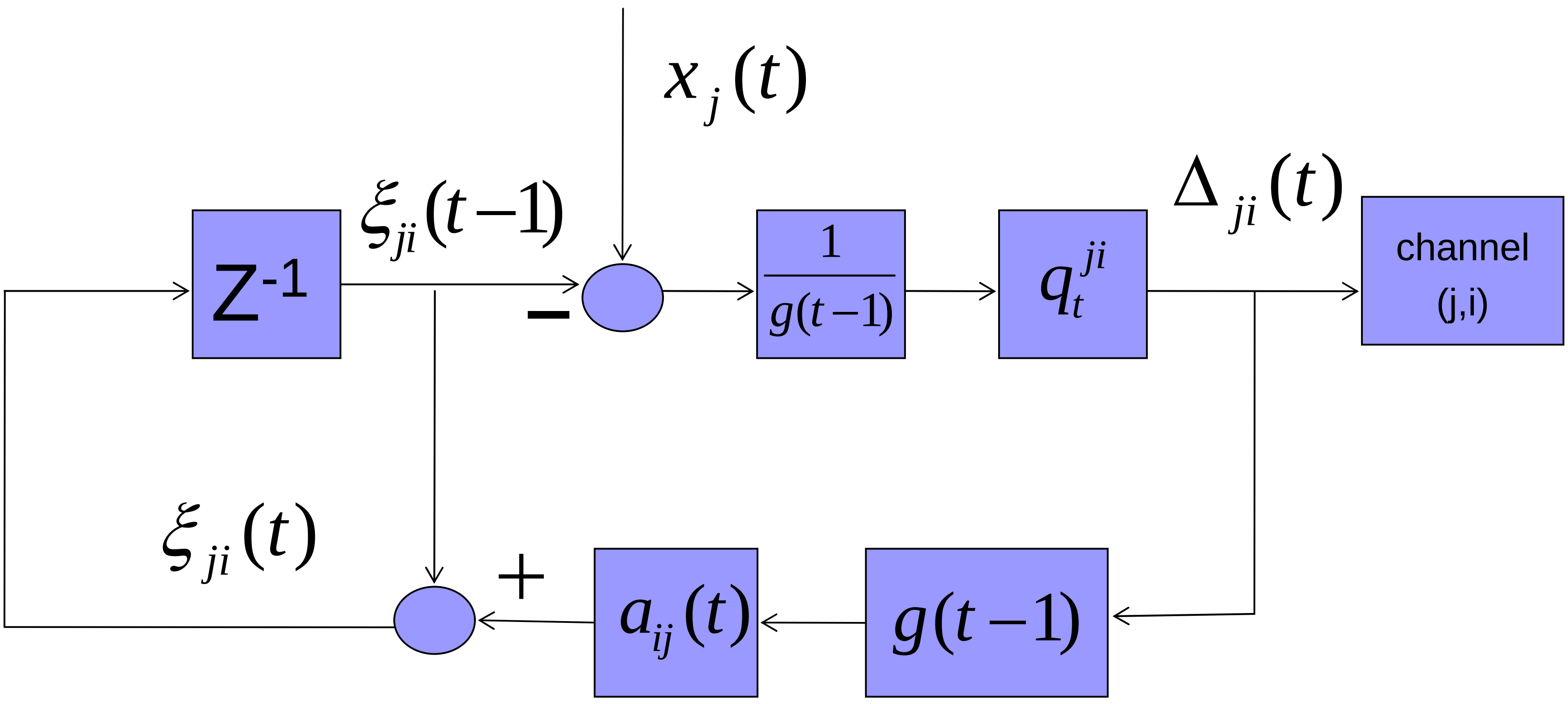
$N_i(t) = \{ j \in V \mid a_{ij}(t) = 1\},\ N_i = \bigcap_{t=1}^{\infty} \bigcup_{k=t}^{\infty} N_i(k)$

$a_{ij}(t) = 1$ means channel (j, i) is active, while

$a_{ij}(t) = 0$ means channel (j, i) is inactive.

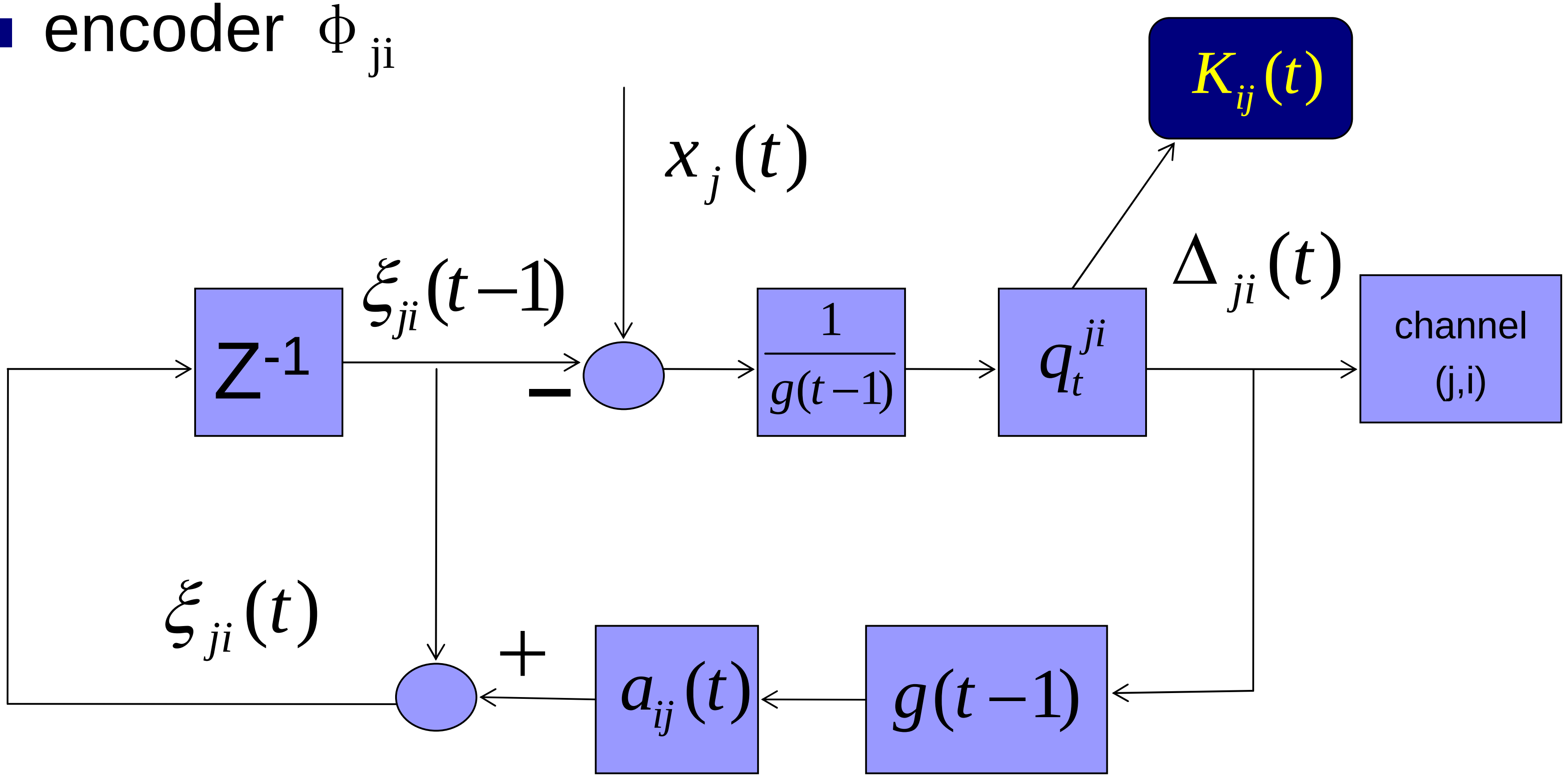
Distributed protocol

- encoder Φ_{ji}



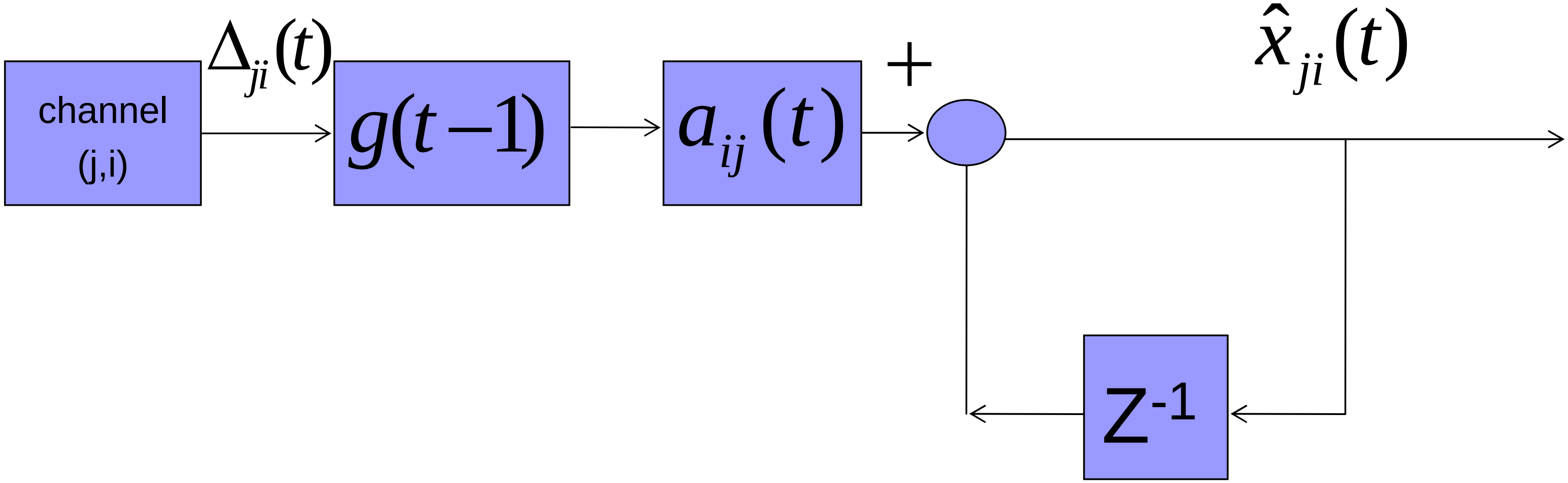
Distributed protocol

- encoder ϕ_{ji}



Distributed protocol

- decoder Ψ_{ji}



Distributed protocol

- protocol

$$u_i(t) = h\sum_{j\in N_i} a_{ij}(\hat{x}_{ji}(t) - \xi_{ij}(t)),\ t = 0,1,\ldots,$$

$$i = 1,2,\ldots,N$$

$$\Downarrow$$

$$\sum_{i=1}^{N} u_i(t) \equiv 0$$

$$\Downarrow$$

$$\sum_{i=1}^{N} x_i(t) \equiv \sum_{i=1}^{N} x_i(0)$$

Main assumptions

A3) $N_i = \bigcup_{t=1}^{\infty} N_i(t),\ i = 1, 2, \ldots, N$

A4) $\max_{1 \le i \le N} |x_i(0)| \le C_x,\ \max_{1 \le i \le N} |\delta_i(0)| \le C_\delta$

A5) There is a constant $d^* > 0$, such that $\max_{1 \le i \le N} \sup_{t \ge 0} d_i(t) \le d^*$

A6) There exist an integer $T>0$ and a real constant $\rho > 0$ such that

$$\inf_{m \ge 0} \lambda_2 \left(\frac{1}{T} \sum_{k=mT+1}^{(m+1)T} L(k) \right) \ge \rho$$

Theorem. Suppose A3)-A6) hold. If

$$h \in \left(0, \frac{1}{2d^*}\right), \ \mu < \frac{1}{\left(1 - \frac{h\rho}{T+1}\right)^{1/2T}}$$

where $\mu = \sup_{t \geq 0}\left[g(t) / g(t+1)\right]$,

and the numbers of quantization levels satisfy

$$K_{ij}(1) \geq \frac{C_x}{g(0)} - \frac{1}{2}, \qquad K_{ij}(2) \geq \begin{cases} \zeta_1(h, g(0), g(1)), a_{ij}(1) = 1, \\ \zeta_2(h, g(0), g(1)), a_{ij}(1) = 0, \end{cases}$$

$$K_{ij}(t+1) \geq \begin{cases} \kappa_{h,\mu} + \dfrac{\mu(2hd^*+1)}{2} - \dfrac{1}{2}, & a_{ij}(t)=1, \\ \kappa_{h,\mu} + \eta_{ij}(t), & a_{ij}(t)=0, \end{cases}$$

$$t = 2, 3, \ldots$$

where

$$\kappa_{h,\mu} = \frac{\sqrt{2N} C_\delta h d^* \mu^2 (1 - \frac{h\rho}{T+1})^{1/2T}}{g(0)} + \frac{\sqrt{2N} h^2 \mu^2 (d^*)^2}{1 - (1 - \frac{h\rho}{T+1})^{1/2T} \mu}$$

$$\eta_{ij}(t) = \frac{g(t-1)}{g(t)}(hd^* + K_{ij}(t)) + \frac{g(t-1)/g(t) - 1}{2},$$ then

$$\limsup_{t\to\infty} \frac{\max_{ij} |x_i(t) - x_j(t)|}{g(t)} \leq \frac{\sqrt{2N} h d^* \mu^2}{1 - \left(1 - \frac{h\rho}{T+1}\right)^{1/2T} \mu}.$$

Finite duration of link failures

- NCS with finite dropouts
  - Xiao, et al., IJNRC, 2009
  - Xiong & Lam, Automatica, 2007
  - Yu, et al., CDC, 2004

A7) There is an integer $T_R > 0$, such that

$$t_{ij}(k+1) - s_{ij}(k) \le T_R, i = 1, 2, \dots, N, j \in N_i.$$

Theorem. Suppose A3)-A7) hold. For any given $K \geq 1$

let

$$\Omega_{K,T_R} = \{(h,\mu) \mid h \in (0, \frac{1}{2d^*}), \mu \in (1, \frac{1}{(1-\frac{h\lambda}{T+1})^{1/2T}}),$$

$$\kappa_{h,\mu} + \frac{\mu(2hd^* + 1)}{2} \leq K + \frac{1}{2},$$

$$\mu^{T_R} K + (\mu h d^* + \kappa_{h,\mu}) \frac{\mu^{T_R} - 1}{\mu - 1} + \frac{\mu^{T_R} - 1}{2} \leq K + 1\}$$

Then

Ω_{K,T_R} is nonempty, and if

selecting h and $g(t)$, such that $(h,\mu)\in\Omega_{K,T_E}$, and

$$K_{ij}(1)=K_{ij}(2)=K,$$

$$K_{ij}(t+1)=\begin{cases}K, & a_{ij}(t)=1,\\ K+1, & a_{ij}(t)=0,\end{cases} \quad t=2,3,\ldots,$$

$$\lim_{t\to\infty} x_i(t)=\frac{1}{N}\sum_{j=1}^{N} x_j(0),\ i=1,2,\ldots,N$$

Note that $\lceil \log_2 5 \rceil = 3$

If the duration of link failures in the network is bounded, then control gain and scaling function can be selected properly， s. t.

3-bit quantizers suffice for asymptotic average-consensus with an exponential convergence rate.

Concluding remarks

Fixed topology: protocol based on difference encoder and decoder

Asymptotic average-consensus with a single-bit exchange

Relationship between asymptotic convergence rate and system parameters

Selecting the control parameters for energy minimization

Time-varying topology: different quantizers for different channels

Adaptive algorithm for selecting the numbers of quantization levels

Special case with jointly-connected graph flow and finite duration of link failures: 3-bit quantizer

Other topics

- Case with time-delay

  S. Liu, T. Li, L. Xie, Distributed consensus for multi-agent systems with communication delays and limited data rate, SIAM J. Control & Optimization, November, 2011.

- Case with partial measurable states

  T. Li, L. Xie, Distributed coordination of multi-agent systems with quantized-observer based encoding-decoding, to appear in IEEE TAC, 2012.

Future topics

- Noisy channels
- Random link failures
- Model uncertainty
- …

Thank you !

Nonlinear coupling between consensus error and quantization error

$$\delta(t+1) = f(\delta(t), e(t))$$

$$e(t+1) = g(\delta(t), e(t))$$

Nonlinear coupling between consensus error and quantization error

$$\delta(t+1) = f(\delta(t), e(t))$$

$$e(t+1) = g(\delta(t), e(t))$$

Adaptive control

$$\varphi(t+1) = f(\varphi(t), \tilde{\theta}(t))$$

$$\tilde{\theta}(t+1) = g(\varphi(t), \tilde{\theta}(t))$$

$$\delta(t+1) = f(\delta(t), e(t))$$

$$\|\delta(t+1)\| \le \phi(h, K, g(t)) \sup_{0 \le k \le t} \|e(k)\|$$

$$e(t+1) = g(\delta(t), e(t))$$

$$\sup_{t \ge 0} \|e(t)\| < \infty$$

Selecting proper h, K, $g(t)$

$$\delta(t) = o(1), \; t \to \infty$$

Lemma. Suppose A1)-A2) hold. For any given

$h \in (0, 2/\lambda_N)$ and $\gamma \in (\rho_h, 1)$

If $2K+1 \geq 2K_1(h,\gamma)+1$

then $$\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(0), \quad i = 1, 2, \ldots, N$$

by taking $g(t) = g_0 \gamma^t$ with

$$g_0 > \max\left\{ \frac{C_x}{K+1/2}, \ \frac{2(\gamma-\rho_h)(C_\delta \gamma + h C_x \lambda_N)}{h\lambda_N} \right\}$$

Parameter selection algorithm

- Selecting a constant $\varepsilon_0 \in (0,1)$
- Selecting the control gain $h \in (0, h_K^*(\varepsilon_0))$

$$h_K^*(\varepsilon_0) = \min\left\{\frac{2}{\lambda_2 + \lambda_N}, \frac{2K\varepsilon_0\lambda_2}{\sqrt{N\lambda_N^2 + 2\lambda_2 d^*\varepsilon_0 + (2K+1)\lambda_2^2\varepsilon_0(1-\varepsilon_0)}}\right\}$$

- Choose $\gamma = 1-(1-\varepsilon_0)h\lambda_2$

Nonlinear coupled inequalities

$$\Omega_K = \left\{(\alpha,\beta) \mid 0<\alpha<\frac{2}{\lambda_2+\lambda_N},\ \rho_\alpha<\beta<1,\ M_1(\alpha,\beta)<K+\frac{1}{2}\right\}$$