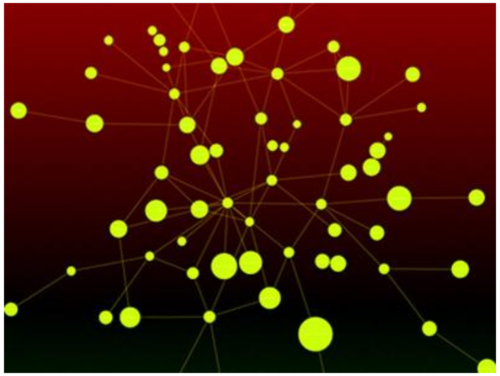
Searching for Networks with Best Possible Synchronizability



Dinghua Shi, Guanrong (Ron) Chen
Wilson W. K. Thong, Xiaoyong Yan
City University of Hong Kong

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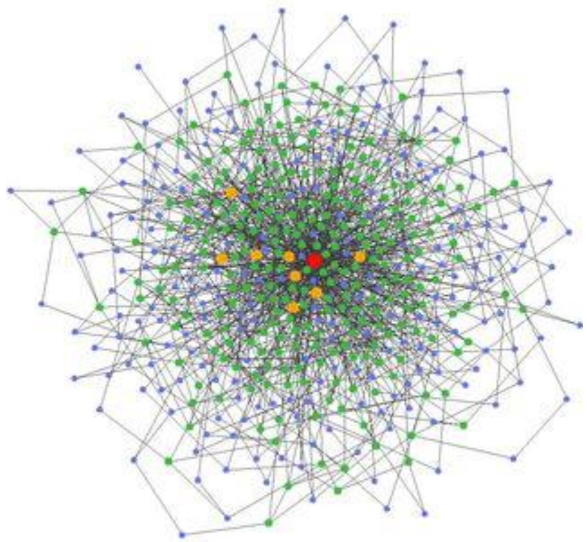


Network Model

Linearly coupled network:

$$\frac{dx_i}{dt} = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1,2,\ldots,N$$

a general assumption is that *f* (.) is Lipschitz

coupling strength $c > 0$ and

coupling matrices (undirected):

$$A = [a_{ij}]_{N\times N} \qquad H(x_j) = \begin{bmatrix} H_1(x_j) \\ H_2(x_j) \\ \vdots \\ H_n(x_j) \end{bmatrix} \qquad \text{e.g.} \quad H = \begin{bmatrix} r_{11} & & & 0 \\ & r_{22} & & \\ & & \ddots & \\ 0 & & & r_{nn} \end{bmatrix}$$

A: If node i connects to node j $(j \neq i)$, then $a_{ij} = a_{ji} = 1$; else, $a_{ij} = a_{ji} = 0$; also, $a_{ii} = 0$

Laplacian matrix: $L = D - A \qquad D = diag\{d_1, \ldots, d_n\} \qquad d_i$ - degree of node i

Network Synchronization

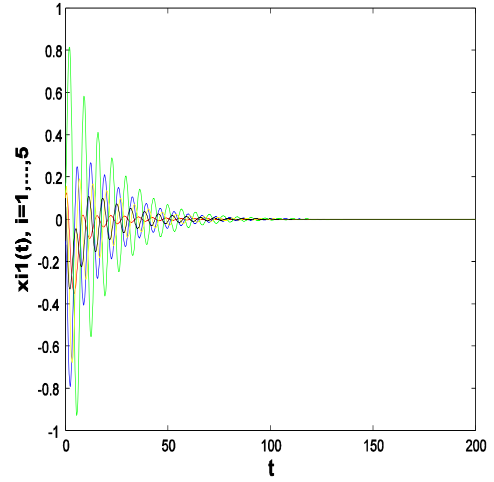
$$\frac{dx_i}{dt} = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1,2,\ldots,N$$

complete state synchronization:

$$\lim_{t\to\infty} \| x_i(t) - x_j(t) \|_2 = 0, \qquad i, j = 1,2,\ldots,N$$

Example:

Synchronized to zero equilibrium



Preliminary

- Variational Equation:

$$d\xi_k/dt = \left([Df(x)] - c\lambda_k I\right)\xi_k, \quad k = 1, \cdots, N$$

where $[Df(x)]$ is Jacobian (e.g., uniformly bounded)
The maximum Lyapunov exponent of the equation is called the *master stability function*.

- Laplacian has zero row-sum

→

Spectrum: $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$

What affects the synchronizability

(i) inner-linking matrix H (fixed)
(ii) outer-linking matrix A (varying)

Synchronizability in terms of (ii):

1. unbounded region (X.F. Wang and G.R. Chen, 2002)

   Spectral gap λ_2, $\quad 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$

2. bounded region (M. Banahona and L.M. Pecora, 2002)

   Eigenratio λ_2 / λ_N, $\quad 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$

3. union of several disconnected regions

   (A. Stefanski, P. Perlikowski, and T. Kapitaniak, 2007)
   (Z.S. Duan, C. Liu, G.R. Chen, and L. Huang, 2007-2009)

What is the idea

Error equation: $\dot{e} = Ae$

 $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ Eigenvalues of L: $\lambda_1 = 0 \quad \lambda_2 = 2 > 0$

One state component: $e_2(t) = e^{-\lambda_2 t} \rightarrow 0 \qquad (t \rightarrow \infty)$

Bigger λ_2 → Faster convergence

$$e_1(t) - e_2(t) \equiv 0, \quad e_2(t) \rightarrow 0, \quad \Rightarrow \quad e_1(t) \rightarrow 0$$

Question (for example)

Given Laplacian:

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 3 & 0 & -1 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & -1 & -1 & 4 & -1 \\ -1 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$$

Q: How to replace 0 and -1 while keeping the connectivity (and all row-sums = 0), such that λ_2 / λ_N = maximum ?

Answer

$$L^* = \begin{bmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 0 & -1 & 3 \end{bmatrix} \rightarrow \quad \lambda_2/\lambda_N = \text{maximum}$$

Observation: **Homogeneous +** Symmetrical

Problem

- With the same numbers of node and edges, while keeping the connectivity, what kind of network has the best possible synchronizability?
- Observation: Large eigenratio → Large spectral gap

$$\max_{A \in A^*} \frac{\lambda_2}{\lambda_N} = \max_{A \in A^*} \left\{ \min_{x^T e=0, x \neq 0} \frac{x^T [D-A] x}{x^T x} \Big/ \max_{x^T e=0, x \neq 0} \frac{x^T [D-A] x}{x^T x} \right\}$$

Such that

$$\sum_{i=1}^{N} d_i = N\bar{k} \quad \text{and} \quad \lambda_2 > 0$$

- Computationally, this is NP-hard

Progress

- **Nishikawa et al., *Phys. Rev. Lett.* 91, 014101(2003) – regular networks with uniform small (node or edge) betweenness [we found: edge betweenness is more important than node betweenness]**
- **Donetti et al., *Phys. Rev. Lett.* 95, 188701(2005) Entangled networks have biggest eigenratios [we found: not necessarily]**
- **Donetti et al., *J. Stat. Mech.: Theory and Experiment* 8, 1742(2006), algorithm based on algebraic graph theory; big spectral gap → big eigenratio [we found: the opposite]**
- **Zhou et al., *Eur. Phys. J. B*. 60, 89(2007), algorithm based on smallest clustering coefficient**
- **Hui, *Ann Oper. Res.*, July (2009), algorithm based on entropy**
- **Xuan et al., *Physica A* 388, 1257(2009), algorithm based on short average path length**
- **Mishkovski et al., *ISCAS*, 681(2010), fast generating algorithm**

Comparison of Synchroniability

☞ **Guan et al., Chaos 18, 013120(2008),** $N = 100$

$r = \lambda_N / \lambda_2$	Ensemble A	Ensemble B	Ensemble C		
	$k_d=6$	$\bar{k}_d=6$	$k_d=6,\ k_r=10$	$k_d=6,\ k_r=20$	$k_d=6,\ k_r=30$
$\bar{r}$	5.881	19.431	77.964	46.603	33.631
$\bar{\lambda}_N$	10.315	13.786	9.661	10.082	10.401
$\bar{\lambda}_2$	1.759	0.745	0.130	0.224	0.318
$\bar{d}$	2.748	2.746	4.747	4.012	3.674

A: Regular networks (with degree-preserved link switching)

B: Random Networks

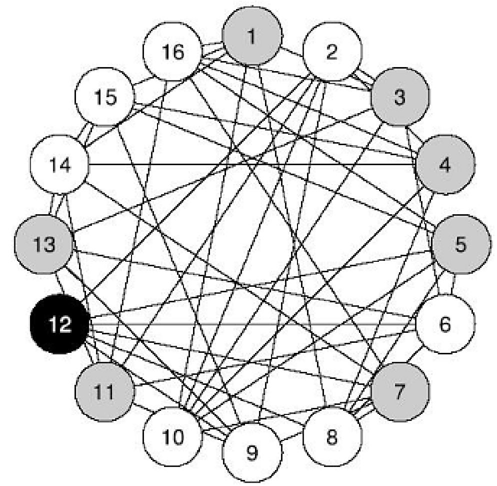
C: Small-world Networks

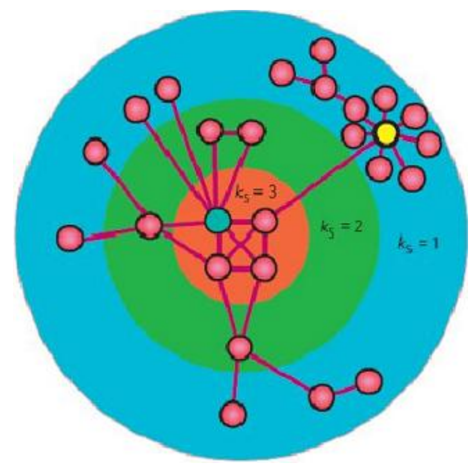
Recent Discoveries

- Biological neural networks, *Phys Reports* 2011 – NN of 16 cells

- *k*-cell, *Nature Phys* 2010 - more important than hubs and high-betweenness nodes in epidemics

- Homogenous networks, *Nature* 2011 - more advantageous for control





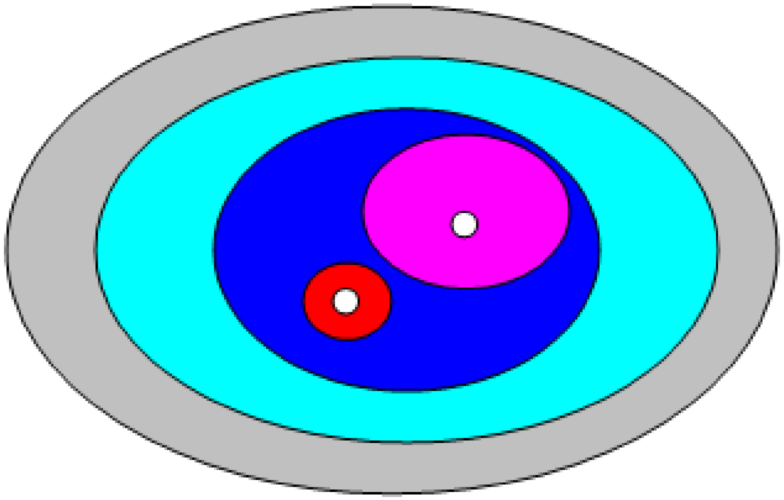
Our New Model

- ☞ **Homogeneity + Symmetry**
- ☞ **Same node degree** $d_1 = \cdots = d_N$
- ☞ **Shortest average path length**
- ☞ **Shortest path-sum** $l_i = \sum_{j \neq i} l_{ij}$ $l_1 = \cdots = l_N$
- ☞ **Longest girth** [girth = shortest loop of a node]

$$g_1 = \cdots = g_N$$

Optimization

☞ **Illustration:**



White：Optimal homogeneous

Grey：networks with same numbers of nodes and edges
Green：degree-homogeneous networks
Blue：networks with maximum girths
Pink：possible optimal networks
Red：near homogenous networks

Our Algorithm (Step A, B, C)

- **Consider the case of average node degree** k = integer
- **Step A -** ***Building Node-degree Homogenous Networks:***
- A1. Initially: given a fully-connected graph of k +1 nodes.

  A2. In each step, if k = odd, then add 2 nodes with $(k–1)/2$ edges connecting to the existing network； if k = even, then add 1 node with k /2 edges connecting to the existing networks.

  A3. Attachment: For non-identical node degrees, use anti-preferential attachment; for identical node degree, use random.

  A4. For every node with degree bigger than the average node degree, randomly select one old edge to do rewiring, so as to obtain a homogenous node degree sequence.

Our Algorithm (continued)

Step B - *Generating Networks with Longest Girth:*

B1. Ordering. List all random node-degree homogeneous networks generated in steps A1-A4, according to their girths in decreasing order (if equal, list their average path-lengths in increasing order; if path-lengths are also equal, list their automorphisms in decreasing order).

B2. Reducing. If two networks have same girth sequences, path-sum sequences and automorphism numbers, then only one is kept for further iterations.

B3. Iterating. Starting from the first network in the above list, return to steps A2-A4, until reaching the end of the list or meeting a pre-set stopping rule, to obtain all networks with the longest girth.

Our Algorithm (continued)

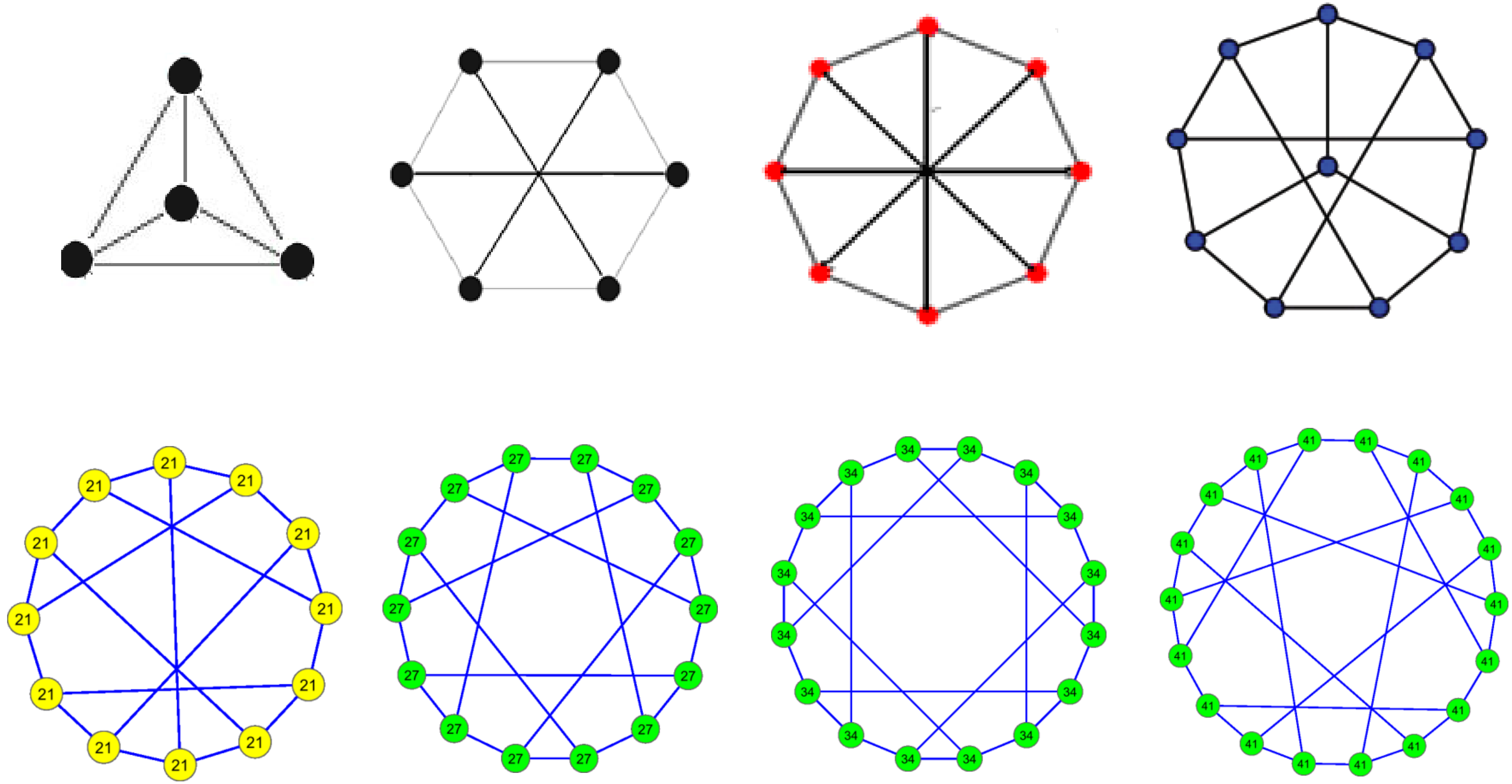
Step C - ***Finding Optimal Homogeneous Networks:***

C1. For each network with the longest girth obtained above, calculate its every node's path-sum, network diameter, and average path-length.

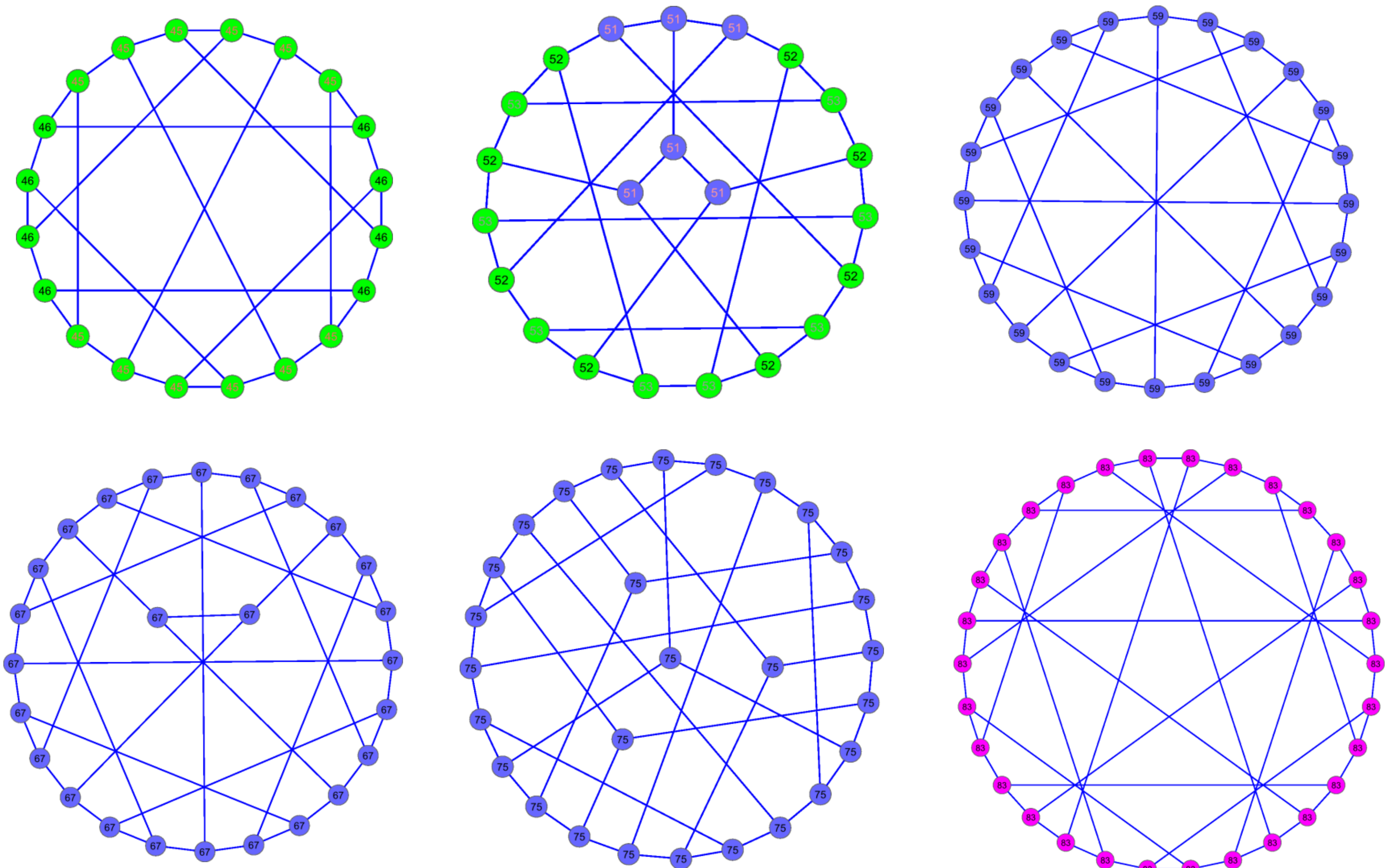
C2. Check the homogeneities of girth and path-sum for every network obtained above. Keep those homogeneous networks with minimum $\ell*$ and those non-homogeneous with $\ell < \ell*/(N-1)$ as candidates.

C3. Calculate the eigen-ratios of all the candidates in C2 and find the biggest one, which is an optimal homogeneous network.

Optimal Networks



Optimal Networks



Optimal Networks

☞ Resulting Networks (first 14 solutions)

Number of nodes (girth, path-sum)	4 (3, 3)	6 (4, 7)	8 (4, 11)	10 (5, 15)	12 (5, 21)	14 (6, 27)	16 (6, 34)
Spectral Gap	4.000	3.000	2.000	2.000	1.468	1.586	1.268
Eigenratio	1.000	0.500	0.369	0.400	0.277	0.264	0.211
Network Diameter	1	2	2	2	3	3	4

Number of nodes (girth, path-sum)	18 (6, 41)	20 (6, x)	22 (x, x)	24 (7, 59)	26 (7, 67)	28 (7, 75)	30 (8, 83)
Spectral Gap	1.268	1.065	1.000	1.000	1.000	1.000	1.000
Eigenratio	0.211	0.189	0.180	0.180	0.174	0.185	0.167
Network Diameter	4	4	4	4	4	4	4

Except:

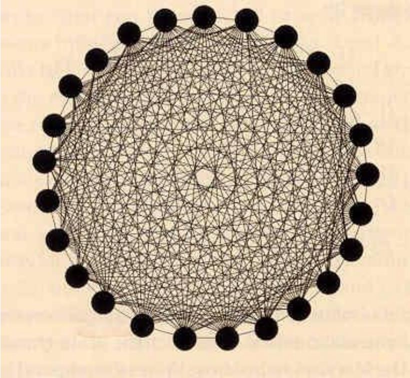
Network with 20 nodes: same girth, but not path-sum

Network with 22 nodes – both girth and path-sum are not

Conclusion

Optimal network topology **(in the sense of having the best possible synchronizability)** **should have**

- **Homogeneity** **(with same node degree)**
- **Symmetrical structure**
- **Shortest path-sum**
- **Longest girth**



Thank You !