# Cluster Synchronization in Dynamical Network Systems with Non-identical Node Dynamics[1]

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# Outlines

# Synchronization

Consider a series of dynamic processes $\{x^1(t), \cdots, x^m(t)\}$ to describe the dynamics of network system with $m$ agents (nodes). (Complete) Synchronization is defined as:

$$\lim_{t \to \infty} \|x^i(t) - x^j(t)\| = 0$$

holds for all $i \neq j$.

# Cluster Synchronization

Divide the set of agents, denoted by $\mathcal{V}$, into disjoint clusters, $\mathcal{C} = \{\mathcal{C}_1, \cdots, \mathcal{C}_K\}$, with the properties:

1. $\mathcal{C}_p \bigcap \mathcal{C}_q = \emptyset$ for each $p \neq q$;
2. $\bigcup_{p=1}^{K} \mathcal{C}_p = \mathcal{V}$.

Define cluster-synchronization via the following aspects:

1. $x(t)$ is bounded;
2. We say that $x(t)$ *intra-cluster synchronizes* if $\lim_{n\to\infty} |x_i(n) - x_{i'}(n)| = 0$ for all $i, i' \in \mathcal{C}_p$ and $p = 1, \cdots, K$;
3. We say that $x(t)$*inter-cluster separates* if $\limsup_{n\to\infty} |x_i(n) - x_j(n)| > 0$ holds for each pair of $i \in \mathcal{C}_k$ and $j \in \mathcal{C}_l$ with $k \neq l$.

# Models of dynamic network system with identical node dynamics: discrete-time case

Start with dynamic network of discrete-time systems

$$x^i(t+1) = \sum_{j=1}^{m} g_{ij} f(x^j(t)),\ i = 1, \cdots, m,\ t = 1, 2, \cdots,$$

where $x^i(t) \in \mathbb{R}$ is the state variables of agent $i$, $f(\cdot)$ is a map and $G = [g_{ij}]_{i,j=1}^{m}$ is a stochastic matrix associated with a (directed) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \cdots, m\}$ and

$$\mathcal{E} = \{(i,j) : \quad g_{ij} \neq 0\};$$

in particular, if $f(\cdot) = id$, then it becomes

$$x^i(t+1) = \sum_{j=1}^{m} g_{ij} x^j(t),\ i = 1, \cdots, m,\ t = 1, 2, \cdots.$$

(Known as the consensus algorithm for discrete-time).

# Models of dynamic network with identical node dynamics: continuous-time case

$$\dot{x}^i = f(x^i(t), t) + \sum_{j=1}^{m} l_{ij}\Gamma x^j(t),\ i = 1, \cdots, m,\ t \geq 0,$$

where $x^i(t) \in \mathbb{R}^n$ is the state variables of agent $i$,
$f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a map, $\Gamma \in \mathbb{R}^{n,n}$ denotes the inner coupling configuration, and $L = [l_{ij}]_{i,j=1}^m$ is the Laplacian matrix associated with a (directed) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \cdots, m\}$ and

$$\mathcal{E} = \{(i, j) : \quad l_{ij} \neq 0\};$$

in particular, if $f(\cdot) = 0$, then it becomes

$$\dot{x}^i = \sum_{j=1}^{m} l_{ij} x^j,\ i = 1, \cdots, m,\ t \geq 0$$

(Known as the consensus algorithm for continuous-time case).

# Models of dynamic network system with inter-cluster non-identical dynamics: discrete-time case

Each cluster has different coupled maps:

$$x^i(t+1) = \sum_{k=1}^{K} \sum_{j \in \mathcal{C}_k} g_{ij} f_k(x^j(t)),\ i = 1, \cdots, m,\ t = 1, 2, \cdots,$$

where the map $f_k(\cdot)$ is identical for agents in the same cluster $k$; for the linear system with $f_\cdot(\cdot) = id$, we add external inputs as follows:

$$x^i(t+1) = \sum_{j=1}^{m} g_{ij} x^j(t) + I_k(t),\ i \in \mathcal{C}_k,\ k = 1, \cdots, K.$$

# Models of dynamic network system with inter-cluster non-identical dynamics: continuous-time case

$$\dot{x}^i = f_k(x^i(t)) + \sum_{j=}^{m} l_{ij}\Gamma x^j,\ i \in \mathcal{C}_k,\ k = 1, \cdots, K,$$

where the map $f_k(\cdot)$ is identical for agents in the same cluster $k$; for the linear system with $f_k(\cdot) = I_k(t)$ if $i \in \mathcal{C}_k$, we add external inputs as follows:

$$\dot{x}^i = \sum_{j=1}^{m} l_{ij}x^j(t) + I_k(t),\ i \in \mathcal{C}_k,\ k = 1, \cdots, K.$$

# Time-varying topologies

Consider a series of graph topologies, $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$, associated with the coupling stochastic matrices $\{G(t)\}_{t=1}^{\infty}$ (discrete-time) or Laplacian matrices $\{L(t)\}_{t\geq 0}$ (continuous-time)

$$\mathcal{E}(t) = \{(i,j):\ g_{ij}(t)(\ or\ l_{ij}(t)) \neq 0\}.$$

Consensus algorithms over time-varying coupling topologies become:

$$x^i(t+1) = \sum_{j=1}^{m} g_{ij}(t) x^j(t) + I_k(t),\ i \in \mathcal{C}_k,\ k = 1, \cdots, K,$$

and

$$\dot{x}^i = \sum_{j=1}^{m} l_{ij}(t) x^j + I_k(t),\ \ i \in \mathcal{C}_k,\ k = 1, \cdots, K.$$

# Cluster synchronization subspace

A clustering $\mathcal{C}$ is defined above as groups of disjoint vertex sets that compose $\mathcal{V}$.

$$\mathbb{S}_{\mathcal{C}} = \{x \in \mathbb{R}^n : \; x_i = x_j \text{ for all } i, j \in \mathcal{C}_p \text{ with } p = 1, \cdots, K\},$$

is named cluster-synchronization subspace. It needs to be invariant through the dynamic network systems.

# Invariance of the cluster synchronization manifold

- The node dynamics $f_i(\cdot)$ (or the input $I_i(t)$) is intra-cluster identical, i.e., $f_i = f_j$ ($I_i(t) = I_j(t)$) for all $i, j \in \mathcal{C}_p$ and all $p = 1, \cdots, K$;
- The coupling matrix has inter-cluster common influence if for each pair of $p$ and $p'$, $\sum_{j \in C_{p'}} l_{ij}$ (or $\sum_{j \in C_{p'}} g_{ij}$) is identical w.r.t. all $i \in \mathcal{C}_p$.

**Lemma 1**

*If both the two conditions above are satisfied, then the cluster-consensus subspace is invariant through the dynamic network systems.*

# Cluster Hajnal diameter

Hajnal diameter is used to measure the distance to synchronization subspace. For a matrix $A$, which has row vectors $A_1, A_2, \cdots, A_n$,

$$\Delta(A) = \max_{i,j} \|A_i - A_j\|$$

By the analog way, for a given clustering $\mathcal{C}$, we define the cluster Hajnal diameter as

$$\Delta_{\mathcal{C}}(A) = \max_{p=1,\cdots,K} \max_{i,j \in C_p} \|A_i - A_j\|$$

for some norm $\|\cdot\|$. It can be seen that $\Delta_{\mathcal{C}}(x) = 0$ is equivalent to the intra-cluster synchronization.

# Cluster spanning tree and cluster scrambling

For a given clustering $\mathcal{C} = \{\mathcal{C}_1, \cdots, \mathcal{C}_K\}$,

- $\mathcal{G}$ has cluster-spanning-trees with respect to (w.r.t.) $\mathcal{C}$ if for each cluster $\mathcal{C}_p$, $p = 1, \cdots, K$, there exists a vertex $v_p \in \mathcal{V}$ such that there exist paths in $\mathcal{G}$ from $v_p$ to all vertices in $\mathcal{C}_p$. We denoted this vertex $v_p$ as the root of the cluster $\mathcal{C}_p$.
- $\mathcal{G}$ is cluster-scrambling (w.r.t. $\mathcal{C}$) if for any pair of vertices $(v_{p_1}, v_{p_2}) \subset \mathcal{C}_p$, there exists a vertex $v_k \in \mathcal{V}$, such that both $(v_k, v_{p_1})$ and $(v_k, v_{p_2})$ belong to $\mathcal{E}$.
- The cluster ergodicity coefficient (w.r.t the clustering $\mathcal{C}$):

$$\mu_{\mathcal{C}}(A) = \min_{p=1,\cdots,K} \min_{i,j \in \mathcal{C}_p} \sum_{k=1}^{N} \min(g_{ik}, g_{jk}).$$

It can be seen that $\mu_{\mathcal{C}}(A) \in [0, 1]$ and $A$ is cluster-scrambling (w.r.t. $\mathcal{C}$) if and only if $\mu_{\mathcal{C}}(A) > 0$.

# Cluster Hajnal inequality

Suppose that stochastic matrices $A$ and $B$ that both have the same dimension and inter-cluster common influence, then

$$\Delta_{\mathcal{C}}(AB) \leq (1 - \mu_{\mathcal{C}}(A))\Delta_{\mathcal{C}}(B).$$

# Linear systems with static topology

Assign inputs as follows:

$$I_i(t) = \alpha_p u(t) \tag{1}$$

where $u(t)$ is a scalar function and $\alpha_1, \cdots, \alpha_p$ are inter-different constants. For linear system,

**Theorem 2**

*Suppose that both $u(t)$ and $\sum_{k=1}^{t} u(k)$ (or $\int_0^t u(s)ds$ ) are bounded, $I(t)$ is defined by (1), and the stochastic matrix $G$ matrix (the Laplacian matrix $L$) is with inter-cluster common influence and cluster-spanning trees. Then for any initial condition $x(0)$, the linear system is bounded and realizes intra-cluster synchronization:*

$$\|x(t)\| \leq M \;\forall\, t, \qquad \lim_{t\to\infty} \|x^i(t) - x^j(t)\| = 0 \;\forall\, i, j \in \mathcal{C}_p,\ p = 1, \cdots, K.$$

*for some $M$.*

# Linear systems with time-varying topologies

The union graph across the time interval $[t, t+T]$ is the graph corresponding to

1. Discrete-time case: $\sum_{s=t}^{t+T} G(t)$
2. Continuous-time case: $\int_t^{t+T} G(s)ds$.

**Theorem 3**

*If there exists $L > 0$ such that the union graph across any L-length time interval has cluster spanning trees, then the linear system realizes intra-cluster synchronization.*

# The cluster synchronized systems

$B = [\beta_{p,q}]_{p,q=1}^{K}$ with

$$\beta_{p,q} = \sum_{j \in \mathcal{C}_q} g_{ij}, \quad i \in \mathcal{C}_p$$

Assume intra-cluster synchronization and let $y^p$ be the synchronized state of the $p$-th cluster and $y = [y^1, \cdots, y^K]^\top$. We have

$$y(t+1) = By(t) + \tilde{\zeta} u(t).$$

**Definition 4**

For a given graph topology and clustering, a property (inter-cluster separation) is said to be generic if for all most every coupling coefficients and almost all initial values, the property holds.

# Generical inter-cluster separation

**Lemma 5**

*Suppose that the coupling matrix has the inter-cluster common influence. Then, for any pair of cluster* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*, either there are no links from* $\mathcal{C}_2$ *to* $\mathcal{C}_1$*; or for each vertex* $v \in \mathcal{C}_1$*, there are at least one link from* $\mathcal{C}_2$ *to* $v$*.*

**Theorem 6**

*Suppose that (1).* $\mathcal{G}$ *satisfies the condition in Lemma 5 w.r.t* $\mathcal{C}$*; (2). the (union) graph has cluster spanning trees. Then the linear system realized (three items of) cluster synchronization generically.*

# Cluster transverse subspace

Let $d = [d_1, \cdots, d_m]^\top$ be a vector with $d_i > 0$ for all $i = 1, \cdots, m$. Define an average state with respect to $d$ in the cluster $\mathcal{C}_k$ as $\bar{x}_d^k = \frac{1}{\sum_{i \in \mathcal{C}_k} d_i} \sum_{i \in \mathcal{C}_k} d_i x^i$. Thus, we denote the projection of $x$ on the cluster synchronization manifold $\mathcal{S}_\mathcal{C}$ with respect to $d$ as: $\bar{x}_d = [\tilde{x}^{1,\top}, \cdots, \tilde{x}^{m,\top}]^\top$ with$\tilde{x}^i = \bar{x}_d^k,\ if\ i \in \mathcal{C}_k$. Then, the variations $x^i - \bar{x}_d^k$ compose the transverse space:

$$\mathcal{T}_\mathcal{C}^d = \{u = [u^{1^\top}, \cdots, u^{m^\top}]^\top \in \mathbb{R}^{mn} :\ u^i \in \mathbb{R}^n,\ \sum_{i \in \mathcal{C}_k} d_i u^i = 0,\ \forall\, k = 1, \cdot$$

We have

$$\mathbb{R}^{mn} = \mathcal{S}_\mathcal{C} \bigoplus \mathcal{T}_\mathcal{C}^d!$$

Intra-cluster synchronization is defined as

$$\mathcal{P}_{\mathcal{T}_\mathcal{C}^d} x = 0 (\to 0)!$$

# Continuous-time case

$$\dot{x}^i = f_p(x^i(t)) + \sum_{j=1}^{m} l_{ij}\Gamma x^j,\; i \in \mathcal{C}_p,\; p = 1, \cdots, K,$$

- Inter-cluster common influence: $\sum_{j \in C_q} l_{ij}$ is same for all $i \in C_p$, $p \neq q$;
- there is $\alpha$ such that $(\xi - \zeta)^\top \Big[ f_k(\xi) - f_k(\zeta) - \alpha\Gamma(\xi - \zeta) \Big] \leq -\delta(\xi - \zeta)^\top(\xi - \zeta)$, for all $k = 1, \cdots, K$.

**Theorem 7**

*If there exists a positive definite diagonal matrix $D$ such that*

$$\Big[ D(L + \alpha I_m) \Big]^s \Big|_{\mathcal{T}_{\mathcal{C}}^d} \leq 0 \tag{2}$$

*holds, then the coupled dynamical system intra-cluster synchronizes.*

# Discrete-time case

$$x^i(t+1) = \sum_{k=1}^{K} \sum_{j \in \mathcal{C}_k} g_{ij} f_p(x^j(t)),\ i \in \mathcal{C}_p,\ p = 1, \cdots, K,$$

- Inter-cluster common influence: $\sum_{j \in C_q} g_{ij}$ is same for all $i \in C_p$, $p \neq q$;
- $\|Df_p(\cdot)\| \leq \kappa$.

**Theorem 8**

*Let $\rho_{\mathcal{T}}^{\mathcal{C}}$ be the maximum eigenvalue (in modulus) of $G$ in the cluster transverse subspace $T_{\mathcal{C}}^d$. If*

$$\kappa \rho_{\mathcal{T}}^{\mathcal{C}} < 1.$$

*then the coupled system intra-cluster synchronizes.*

# Chaos cluster synchronizibility

**Theorem 9**

*For the coupled system (both continuous- and discrete-time cases), suppose that the common inter-cluster coupling condition holds. For some $\alpha > 0$ (or $\kappa > 1$), the sufficient conditions can hold for some realization of coupling coefficients if and only if the graph $\mathcal{G}$ has cluster spanning trees.*

On the contrary, if the graph does not have cluster spanning trees, some cluster will have two disjoint sub-groups. This leads that intra-cluster synchronization cannot be realized.

# Inter-cluster separation

It should be assumed that the differences between $f_k$ for clusters are distinct, which can guarantee that the trajectories are apparently distinguishing even if each cluster synchronizes.

# Undirected graph case

Let us consider the case of undirected graphs. Thus, intra-cluster synchronization can be achieved (for some realization of couplings) if and only if all vertices in the same cluster belongs to the same largest connected component in the graph $\mathcal{G}$. In summary, the following two conditions play the key role in cluster synchronization.

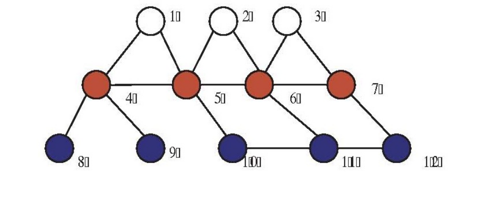
1. Common inter-cluster edges for each vertex in the same cluster;
2. Communicability for each pair of vertices in the same cluster.

# Schemes to clustering synchronize

A cluster is said to be communicable if for each pair of vertices in this cluster, there exists at least one path between them. The paths between vertices are composed of edges either inter-cluster or intra-cluster.

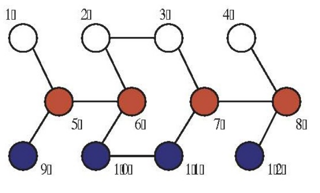
1. A self-organized cluster is a communicable cluster that is communicable only via intra-cluster paths but not communicable only via inter-cluster paths;
2. A driven cluster is a communicable cluster that is communicable only via inter-cluster paths but not communicable only via intra-cluster paths;
3. A mixed cluster is a communicable cluster that is communicable either only via intra-cluster paths or only via inter-cluster paths;
4. A hybrid cluster is a communicable cluster that is communicable neither only via intra-cluster paths nor only via inter-cluster paths.

# Example I



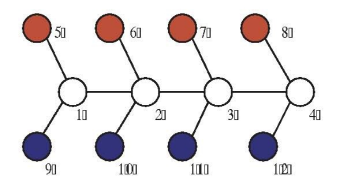

- the white cluster (vertex set $1-3$) is driven since they has no intra-cluster edges;
- the red cluster (vertex set $4-7$) is mixed since each pari of vertices can access each other via only inter- or intra- edges;
- the blue cluster is driven since each pair of vertices can access each other via only the inter-cluster edges but can not communicate only via intra-cluster edges.

# Example II




- Each cluster of the white and blue clusters (vertex sets $1 - 4$ and $9 - 12$) is driven since each pair of vertices can access each other only via inter-cluster edges but only has a single intra-cluster edges;
- the red cluster (vertex set $5 - 8$) is recognized as a hybrid cluster since the sets of inter- or intra-cluster edges are both necessary for communication between each pair of vertices.

# Example III

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- Each The red and blue clusters (vertex sets $5-8$ and $9-12$) are all driven since they do not have intra-cluster edges;
- The white cluster (vertex set $1-4$) is self-organized since each pair of vertices can communicate via only the intra-cluster edges but can not if removing the intra-cluster edges.

# Case study

Consider two graph models:

- $p$-nearest- neighborhood regular graph: The graph has $N$ nodes, ordered by $\{1, \cdots, N\}$. Each node $i$ has $2r$ neighbors:$\{(i+j) \mod N : j = \pm 1, \cdots, \pm r\}$, where mod denotes modular operator. We divide the nodes into $K$ groups: $\mathcal{C}_k = \{i : i \mod K = k\}, k = 0, \cdots, K-1$, where $N \mod K = 0$;
- bipartite random graph: We divide $N$ (an even integer) nodes into two groups and each group has $N/2$ nodes. Each node has $m$ neighbors, among which there are $s < m$ neighbors in the same group and the remaining in another group. The neighbors are chosen with equal probability.

# Dynamic network system

There are two groups and the equations are:

$$\dot{x}^i = \sum_{j=1}^{m} l_{ij} x^j(t) + I_p(t),\ i \in \mathcal{C}_p,\ p = 1, 2.$$

$$I_p(t) = \alpha_p sin(t), p = 1, 2.$$

Measurements:

- Intra-cluster synchronization:

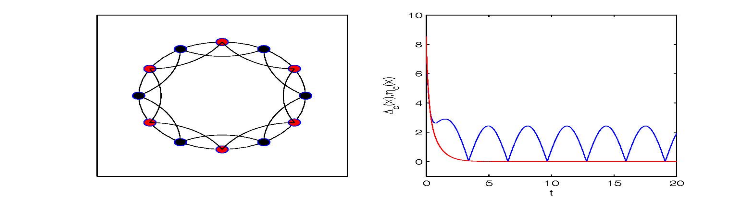
$$\Delta_{\mathcal{C}}(x(t)) = \max_{p} \max_{i,i' \in \mathcal{C}_p} |x_i(t) - x_{i'}(t)|$$

- Inter-cluster separation:

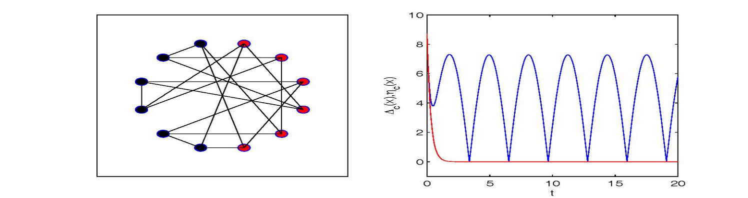
$$\eta_{\mathcal{C}}(x(t)) = \max_{i \in \mathcal{C}_p, j \in \mathcal{C}_q, p \neq q} |x_i(t) - x_j(t)|$$

# Simulation results

Graph model 1:




Graph model 2:



# Nonlinear node dynamics

Three graphs are mentioned above. The equations are:

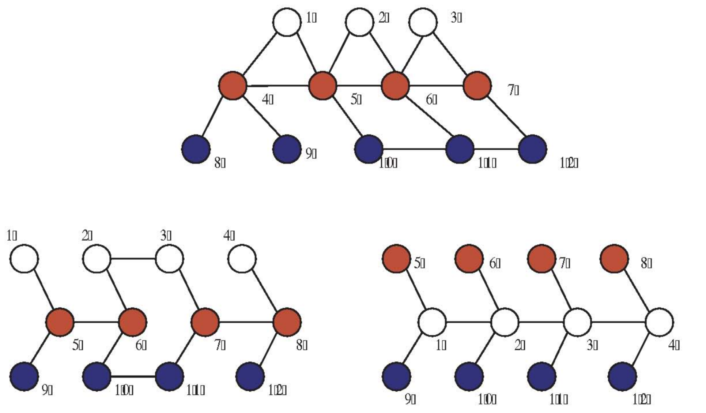
$$\dot{x}^i = f_p(x^i(t)) + \sum_{j=}^{m} l_{ij}\Gamma x^j,\ p \in \mathcal{C}_p,$$

with

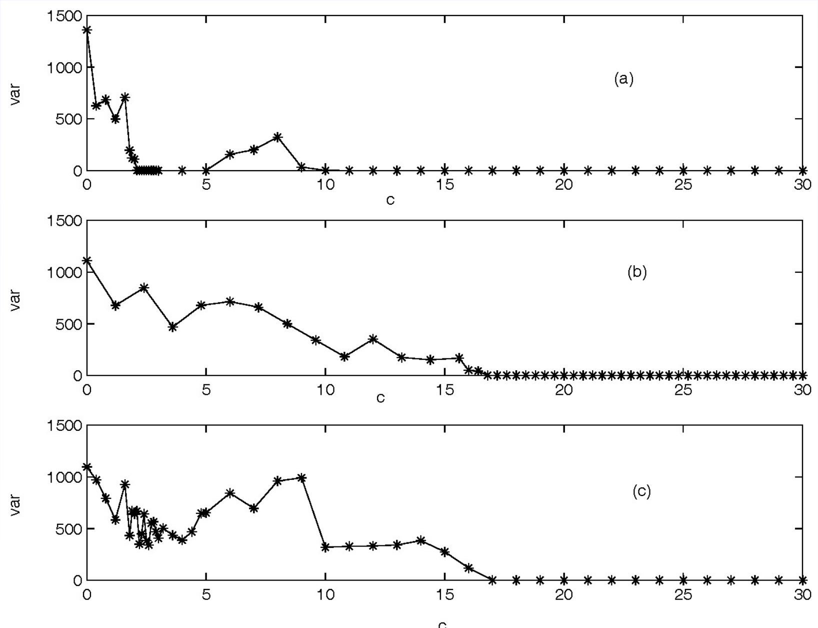
$$f_k(u) = \begin{cases} 10(u_2 - u_1) \\ \dfrac{8}{3}u_1 - u_2 - u_1u_3 \\ u_1u_2 - b_ku_3 \end{cases}$$

for $b_1 = 28$ for the white cluster, $b_2 = 38$ for the red cluster, and $b_3 = 58$ for the blue cluster.

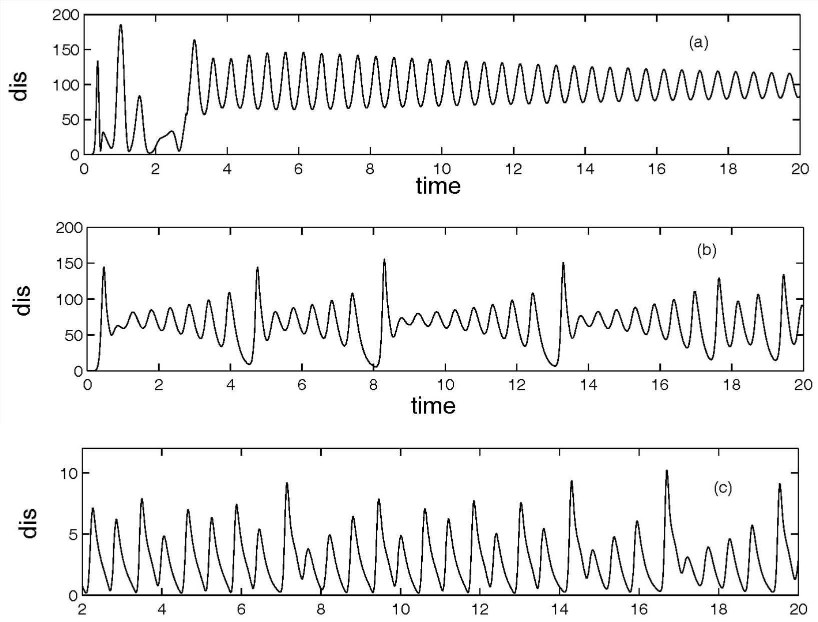
# Graphs



# Simulation Results: intra-cluster synchronization



# Simulation Results: inter-cluster separation



# Conclusions

- Cluster synchronization is defined by two aspects: intra-cluster synchronization and inter-cluster separation;
- Cluster synchronization problem is transformed to the stability analysis of the cluster consensus subspace under inter-cluster common influence condition.
- The separation among states in different clusters are guaranteed by inter-cluster non-identical node dynamics.
- The communicability between agents in the same cluster (possibly via intra-cluster or inter-cluster links ) is a doorsill for the complete intra-cluster synchronization.
- Two schemes to realize intra-cluster synchronization: self-organization and driving.
- The latter scheme implies that cluster synchronization can be realized in a non-clustered networks.

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# Thanks!
# Questions?