Analysis and Intervention of A Class of Multi-Agent Systems

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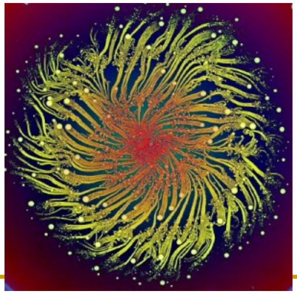
Outline Of My Talk

- Multi-agent Systems
- Analysis: Synchronization
- Intervention: Leader-Follower Model
- Concluding Remarks

Multi-Agent Systems (MAS)

- **Agents in microcosmic level**
  - **Distributed/Autonomous**
  - **Local/Global interactions**
  - **Neighbors may be dynamic**
  - **May have no physical connections**
- **Collective behavior in macroscopic level**

  e.g., clustering, phase transition, pattern, formation, swarm intelligence, consensus, aggregation, …



Bacteria Colony

Ant Colony

Stock Market

Robot Football

From Local Interaction Rules
to
Collective Behavior

（A Basic Model）

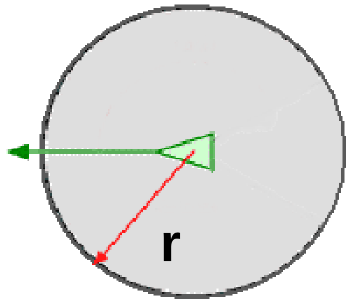
Basic Assumptions

Each agent

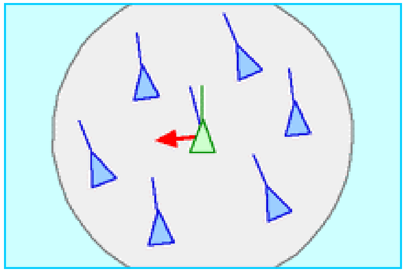
- **makes decision according to local information;**
- **has the tendency to behave as other agents do in its neighborhood.**

Multi-Agent Model

http://angel.elte.hu/~vicsek/



A bird's Neighborhood



Alignment: steer towards the average heading of neighbors

$x_i(t)$: position of agent i ($x_i(t)$ is a 2D vector)

$\theta_i(t)$: **heading of agent i**

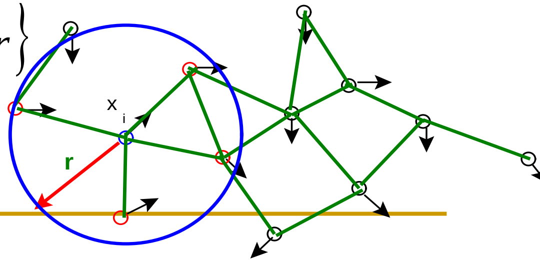
v: the constant speed of birds

r: radius of neighborhood

$$x_i(t+1) = x_i(t) + v(\cos\theta_i(t), \sin\theta_i(t))^\tau$$

$$\theta_i(t+1) = \arctan\left\{\frac{\sum_{j\in N_i(t)} \sin\theta_j(t)}{\sum_{j\in N_i(t)} \cos\theta_j(t)}\right\}$$

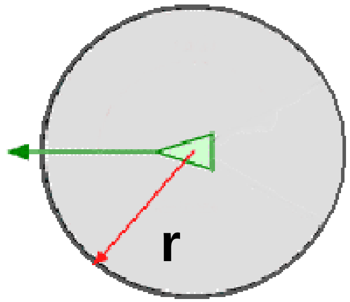
$$N_i(t) = \left\{j : \|x_i(t) - x_j(t)\| < r\right\}$$



Neighbor Graph

Multi-Agent Model

http://angel.elte.hu/~vicsek/



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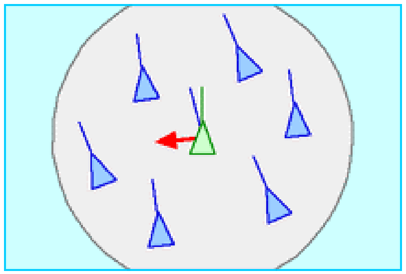

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$$x_i(t+1) = x_i(t) + v(\cos\theta_i(t), \sin\theta_i(t))^\tau$$

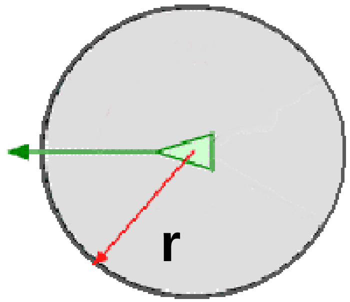
$$\tan\theta(t+1) = \widetilde{P}(t)\tan\theta(t)$$

where the weighted adjacency matrix $\widetilde{P}(t) = \{\widetilde{p}_{ij}(t)\}$ is defined as

$$\widetilde{p}_{ij}(t) = \begin{cases} \dfrac{\cos\theta_j(t)}{\sum_{j\in N_i(t)}\cos\theta_j(t)} & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

Multi-Agent Model

http://angel.elte.hu/~vicsek/



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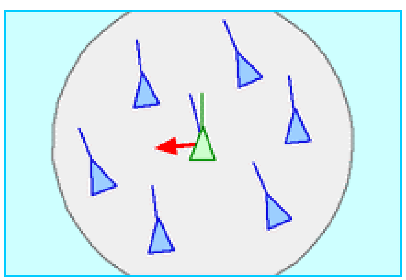

Alignment: steer towards the average heading of neighbors

$x_i(t)$ **: position of agent** i **(**$x_i(t)$ **is a 2D vector)**

$\theta_i(t)$ **: heading of agent** i

v**: the constant speed of birds**

r**: radius of neighborhood**

This model can

— be applied to investigate properties of non-equilibrium systems

—be regarded as a simplified Boid model for flocking behavior

—exhibit some kind of cooperative behavior called synchronization

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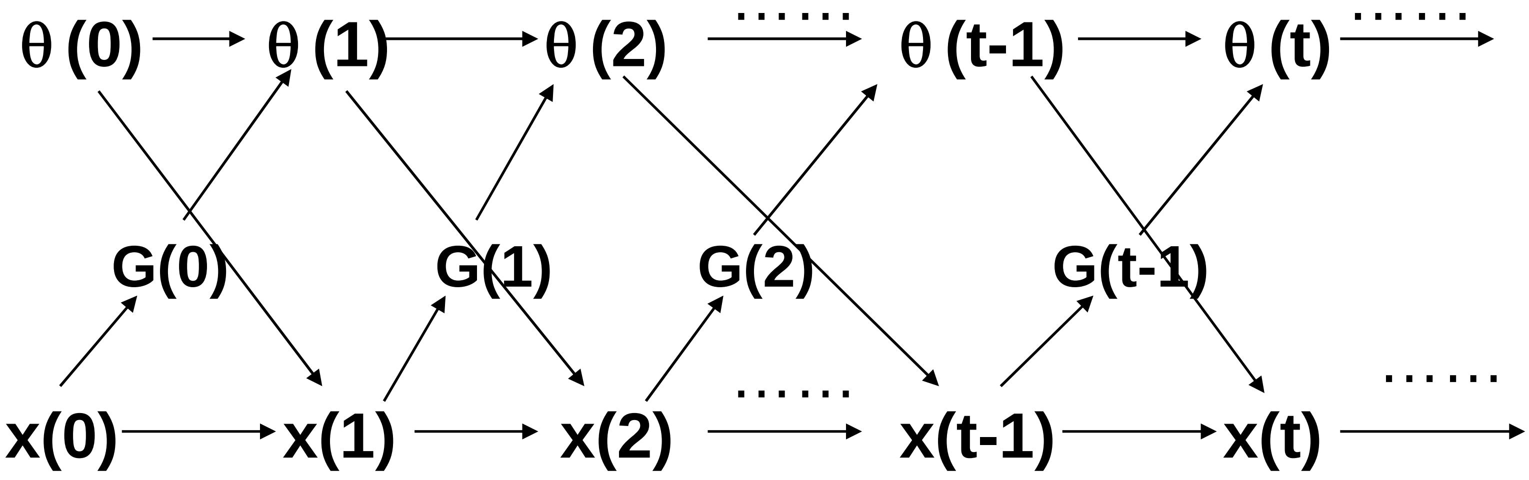
- Analysis: Synchronization

Q: How do we analyze the synchronization behavior?

Interaction and Evolution

Interaction between positions and headings



(1) Positions and headings are strongly coupled

(2) Neighbor graphs may change with time

(3) The dynamical behavior of all agents is determined by the initial states of all agents and model parameters *v* and *r*

Towards a Mathematical Theory

A Stochastic Framework

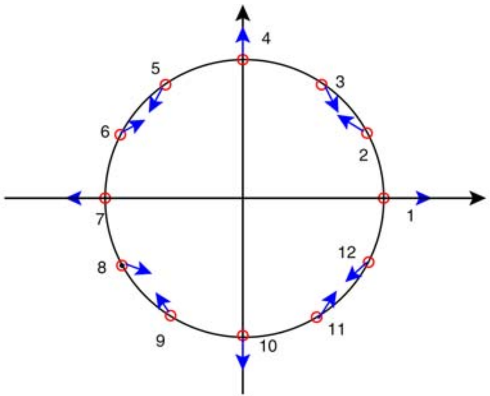
The initial position and heading of all agents are mutually independent, with positions uniformly and independently distributed in the unit square, and headings uniformly and independently distributed in $(-\pi+\varepsilon, \pi-\varepsilon)$ with $\varepsilon \in (0, \pi)$.

Remark: In the stochastic framework, the restriction $(-\pi+\varepsilon, \pi-\varepsilon)$ on the headings can be replaced by $(\alpha, \alpha+2\pi-\varepsilon)$ with any constant α, on which the uniform distribution assumption of the headings can also be replaced by any other distributions. Moreover, $\varepsilon = 0$ may lead to difficulties in guaranteeing synchronization, the following counterexample will give us some clue on this.

Example: $n = 12, 0 < v \le 0.1, r = 0.8$

$$(x_i(0), y_i(0)) = \left(\cos\frac{(i-1)\pi}{6}, \sin\frac{(i-1)\pi}{6}\right);$$

$$\theta_i(0) = \left\{[16 - i + 3\cdot(-1)^i]\frac{\pi}{6}\right\} \bmod (2\pi), \quad 1 \le i \le 12.$$



Random Geometric Graph

Geometric Graph ***G*(*V*, *r*) :**

$$V = \{1, 2, \cdots, n\},$$

$$E = \{(i, j) : \|x_i - x_j\| < r,\ i, j \in V\}$$

Random geometric graph $G(V, r_n)$ **:**

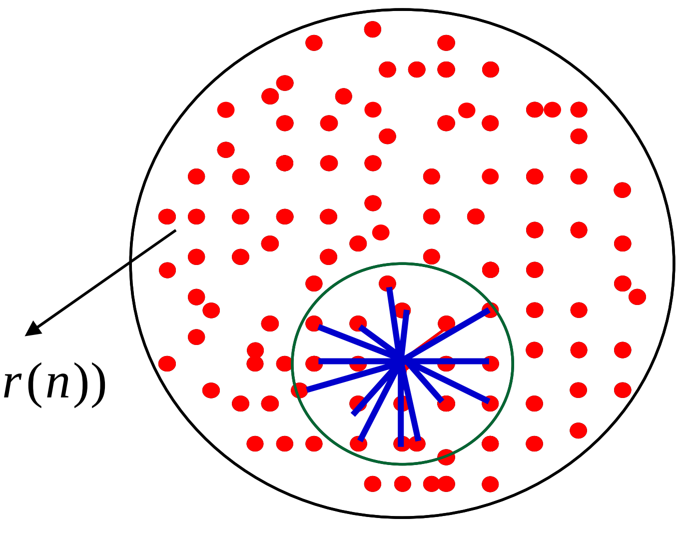
$\{x_i, 1 \le i \le n\}$ **are i.i.d. random vectors.**

M.Penrose, *Random Geometric Graphs*, Oxford University Press,2003.

Connectivity of Random Geometric Graphs

The graph $G(n, r(n))$ **with** $r(n) = \sqrt{\frac{\log n + c(n)}{\pi n}}$ **is connected with probability one as** $n \to \infty$ **if and only if** $c(n) \to \infty$.

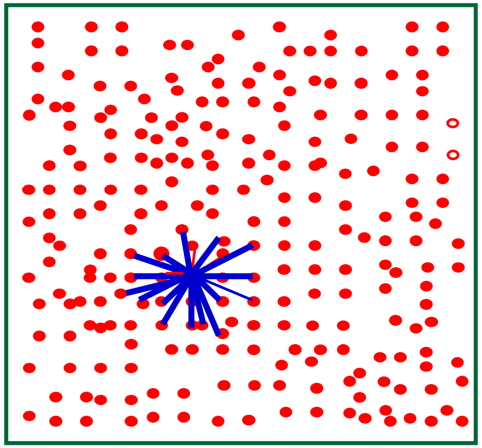
But this result is for static graphs !



(P.Gupta, P.R.Kumar,1998)

Connectivity of Nearest-Neighbor Graph

NNG $G(n, M_n)$: Each node is connected with the M_n nearest neighbors.



Theorem (F.Xue, P.R.Kumar, 2004)

For $G(n, M_n)$ to be asymptotically connected,

$$M_n = \Theta(\log n)$$

neighbors are necessary and sufficient.

Some Key Points
in Theoretical Analysis

- **How to deal with changes of the neighbor graphs ?**
- **How to estimate the synchronization rate?**
- **How to deal with matrices with increasing dimension ?**
- **How to deal with the inherent nonlinearity?**

Some Basic Concepts

Adjacency matrix: $A=\{a_{ij}\},\ a_{ij} = \begin{cases} 1 & \textbf{If } \boldsymbol{i \sim j} \\ 0 & \textbf{Otherwise} \end{cases}$

Degree: $d_i, i=1,\cdots,n,\ d_{\max} = \max_i d_i,\ d_{\min} = \min_i d_i$

Degree matrix: $T = diag(d_1, d_2, \cdots, d_n)$

Average matrix: $P = T^{-1}A$

Laplacian: $L = T - A$

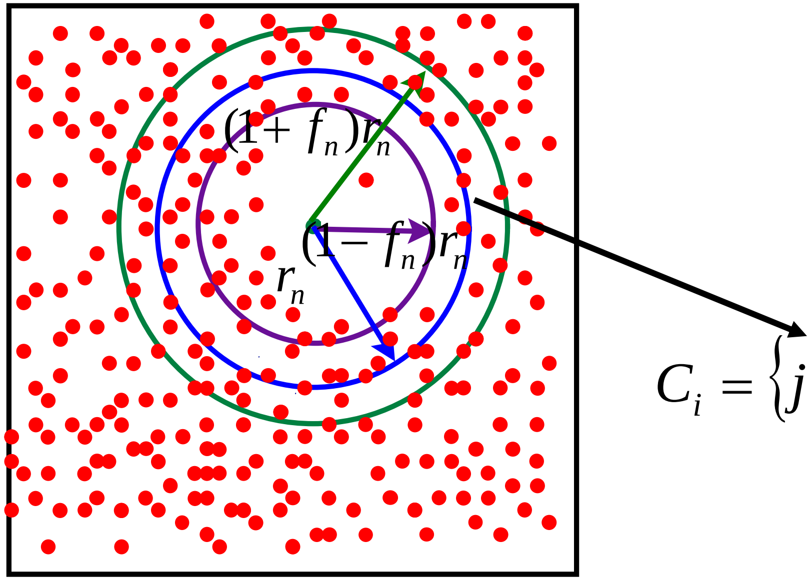
Normalized Laplacian: $\mathcal{L} = T^{-1/2} L T^{-1/2}$

Spectral gap: $\bar{\lambda} = \max \{|1-\lambda_1|, |\lambda_{n-1} - 1|\}$

Dealing with changes of neighbors

(1) Stability of time-varying systems (L. Guo, 1994)

(2) Estimating the number of agents in a ring



$$C_i = \left\{ j : (1-f_n)r_n < \left\| x_j - x_i \right\| < (1+f_n)r_n \right\}$$

Some Key Points in Theoretical Analysis

- **How to deal with changes of the neighbor graphs?**
- **How to estimate the synchronization rate?**
- **How to deal with matrices with increasing dimension?**
- **How to deal with the inherent nonlinearity?**

Synchronization Rate:

The estimation of spectral gap

Normalized Laplacian: $T^{-1/2}LT^{-1/2}$

Spectrum : $0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}$

Spectral gap: $\bar{\lambda} = \max\left(|1-\lambda_1|, |1-\lambda_{n-1}|\right)$

Rayleigh quotient

$$\lambda_1 = \inf_{z \perp T1_n} \frac{\sum_{i\sim j}(z_i - z_j)^2}{\sum_{j\in V} z_j^2 d_j} \qquad \lambda_{n-1} = \sup_{z} \frac{\sum_{i\sim j}(z_i - z_j)^2}{\sum_{j\in V} z_j^2 d_j}$$

$$\Rightarrow \quad \lambda_1 \ge \frac{n d_{\min}}{d_{\max}^2 m l}$$

For random geometric graph with large *n*,

$$d_{\max} = n\pi r_n^2(1+o(1)), \; d_{\min} = \frac{n\pi r_n^2}{4}(1+o(1)).$$

Synchronization Rate:

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For $G(n,r(n))$ with large n

$$\lambda_{n-1} \le 2\left(1 - \frac{1}{12\pi}(1+o(1))\right)$$

$$\lambda_1 \ge \frac{r_n^2}{288\pi}(1+o(1)).$$

$$\Longrightarrow \quad \bar{\lambda} \le 1 - \frac{r_n^2}{144}(1+o(1))$$

Analysis of Matrices with Increasing Dimension

Estimation of double-array martingales

(Guo,1990, Huang&Guo,1990)

$$\max_{1\le m\le n}\max_{1\le k\le n}\left\|\sum_{j=1}^{m} f_j(k,n)w_{j+1}\right\|\le \frac{3C_w}{4}S_n+3\log n,\quad a.s.$$

where $S_n=\max_{1\le k\le n}\sum_{j=1}^{n} f_j^2(k,n),\quad C_w=\sup_{1\le k\le n} E\left(w_{j+1}^2\middle|F_j(n)\right).$

Moreover, if $S_n\ge 4\,C_w^{-1}\log n$, **then we have**

$$\max_{1\le m\le n}\max_{1\le k\le n}\left\|\sum_{j=1}^{m} f_j(k,n)w_{j+1}\right\|\le 3\sqrt{C_wS_n\log n}\quad a.s.$$

Dealing with Inherent Nonlinearility

By applying system structure and estimation of some characteristic on the initial states, we can deal with the inherent nonlinearility.

Under some mild condition on the speed and radius, we have for large n,

$$\sup_{t \geq 1} \| \tilde{P}(t) - P(t) \| = O\left(\sqrt{\frac{\log n}{n r_n^2}} \right), \quad a.s.$$

Result 1: High Density Implies Synchronization

For any given model parameters $v > 0$ and $r > 0$, the MAS will synchronize almost surely, when the number of agents n is large .

Result 2: High density with short distance interaction

Intuitively, when the number of agents increases, the interaction radius can be allowed to decrease with the number of agents, so does the moving speed. Under such a situation, what conditions are needed for synchronization?

Theorem: If the neighborhood radius and the speed satisfy the following condition:

$$\left(\frac{\log n}{n}\right)^{1/6} << r_n << 1, v_n = O\left(\frac{r_n^6 \sqrt{n}}{(\log n)^{3/2}}\right).$$

Then for large population, the MAS will again synchronize almost surely.

Result 3: Synchronization of the three dimensional model

Three dimensional model seems more practical to simulate the motion of animals, such as birds and fish.

Position: $$X_i(t+1) = X_i(t) + v \begin{pmatrix} \cos\theta_i(t+1)\cos\eta_i(t+1) \\ \cos\theta_i(t+1)\sin\eta_i(t+1) \\ \sin\theta_i(t+1) \end{pmatrix}$$

Heading: $$\theta_i(t+1) = \arctan \frac{\sum\limits_{j\in\mathcal{N}_i(t)} \sin\theta_j(t)}{\left\{\Big(\sum\limits_{j\in\mathcal{N}_i(t)} \cos\theta_j(t)\cos\eta_j(t)\Big)^2 + \Big(\sum\limits_{j\in\mathcal{N}_i(t)} \cos\theta_j(t)\sin\eta_j(t)\Big)^2\right\}^{1/2}}$$

$$\eta_i(t+1) = \arctan \frac{\sum_{j\in\mathcal{N}_i(t)} \cos\theta_j(t)\sin\eta_j(t)}{\sum_{j\in\mathcal{N}_i(t)} \cos\theta_j(t)\cos\eta_j(t)}$$

A key issue in the synchronization analysis is to deal with the decaying rate of the non-homogeneous term.

Results 3: Synchronization of the three dimensional model

Theorem: **Under the stochastic framework, if the speed and the radius satisfy**

$$\sqrt[7]{\log^3 n / n} << r_n << 1,\ v_n = O\left(r_n^3\right),$$

Then the three dimensional model will synchronize almost surely for large *n*.

Theorem: **For any given model parameters *v*>0 and *r*>0, the three dimensional model can reach synchronization when the number of agents is large.**

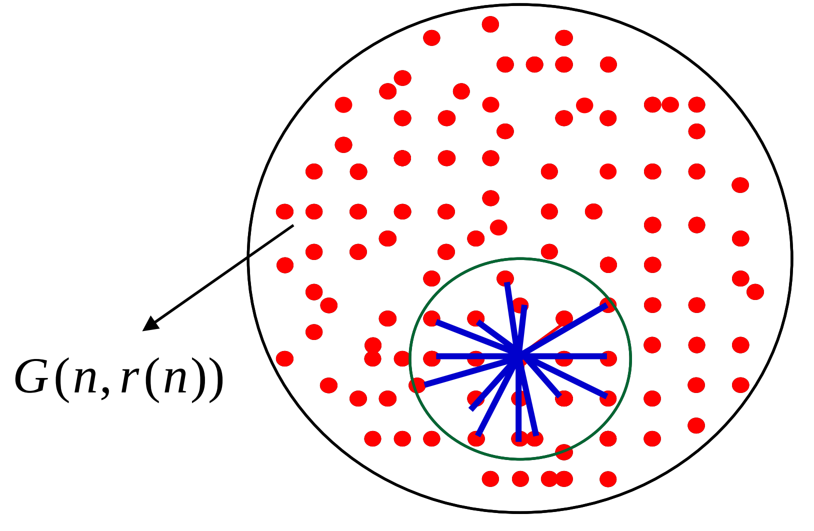
Submitted to IEEE Trans. Auto. Control, 2010

Result 4: The smallest possible interaction radius for synchronization

The graph $G(n, r(n))$ **with** $r(n) = \sqrt{\dfrac{\log n + c(n)}{\pi n}}$ **is connected**

with probability one as $n \to \infty$ **if and only if** $c(n) \to \infty$.

But this result is for static graphs !



(P.Gupta, P.R.Kumar,1998)

Result 4: The smallest possible interaction radius for synchronization

Theorem Suppose that the n agents are initially i.i.d. in $[0, 1]^2$, and that the radius satisfies

$$\lim_{t\to\infty}\left(n\pi r_n^2 - \log n\right) = \infty.$$

If $v_n = o\left(r_n (\log n)^{-1} n^{-2}\right)$, then the simplified system will synchronize w.h.p. for all initial headings and sufficiently large n.

Remark: If the radius satisfies

$$\lim_{n\to\infty}\left(\pi n r_n^2 + 3\log\log n - \log n\right) = -\infty,$$

then w.h.p. there exist some initial headings such that the system can not reach synchronization for any speed.

G. Chen, Z. X. Liu, L. Guo, Accepted by SIAM 2012.

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If $v_n = o\left(r_n (\log n)^{-1} n^{-2}\right)$, then the simplified system will

In probability sense, the supercritical connectivity radius of G_0 can be regarded as the smallest possible radius for synchronization.

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Result 5: Synchronization of MAS with M-Nearest-Neighbor Rule

Experiments show that in some biological systems, the agents interact with a fixed number of neighbors.
(see, M. Ballerini, et al, PNAS, 2008)

Theorem: If the number of agents' neighbors is proportional to the population size n, then for any speed v, the system will synchronize almost surely provided that n is large enough.

C. Chen. G. Chen, L. Guo, CCC, 2011

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Intervention of Multi-Agent Systems

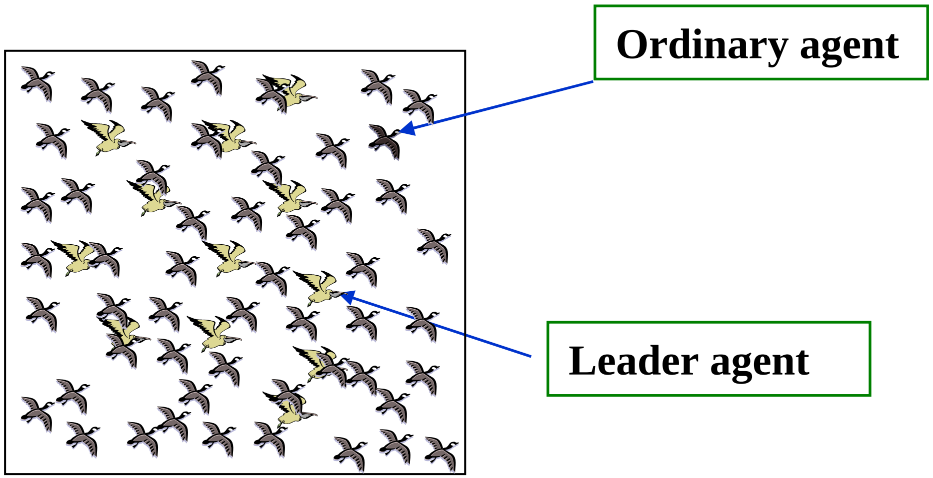
Why we need to intervene the MAS?

- **For the MAS, the dynamical behavior of the agents is determined by the initial states and model parameters.**
- **The behavior resulting from self-organization may not be what we expect.**
- **In many practical situation, we need to guide the system to the desired behavior.**
- **Adding "information" agents (called leaders) may be a feasible and easily manipulated way.**

Intervention of Multi-Agent Systems

Key features of the leader-follower model:

- Not to change the local rule of the existing agents;
- Add some (not very few) "information" agents – called "leaders", to control or intervene; the existing agents treat them as ordinary agents.
- What we can control is the proportion/number of leaders.
- Open loop intervention



Intervention by Leaders

Couzin et al., Nature, Vol 433, p513-515, 2005

- **In some biological systems, a small number of agents have pertinent information about the destination, they can help induce the whole group.**
- **The larger the population size, the smaller the proportion of leaders is needed to guide the system.**

N. C. Makris, et al., Science, Vol. 323, P1734-1737, 2009

Experiments on vast oceanic fish shoals show that small sets of leaders significantly influence the actions of much larger groups.

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How many leaders are needed for the desired behavior?

Intervention by Leaders

To answer the above issue, we focus on the intervention of the Vicsek model, and proved the following results:

Theorem: **Let the proportion of leaders α_n satisfies**

$$\alpha_n = \frac{M_n}{n} \geq C \sqrt[4]{\frac{\log n}{n}},$$

where n is the size of the population. Then all agents will move with the same expected direction eventually.

Remark: Suppose that the model parameters and the proportion of leaders satisfy

$$\sqrt[4]{\frac{\log n}{n}} << r_n, \ v_n \leq \frac{r_n}{2048}, \ \alpha_n = O(v_n r_n^3)$$

Then the leaders can not guide the followers to the expected direction almost surely when the population size is large enough.

Z.X.Liu, J. Han, X. M. Hu, Automatica, 2011

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Concluding Remarks

In this talk,

- We presented synchronization analysis of MASs under stochastic framework with large population
- Our conditions for synchronization only depend on the initial states and model parameters without resorting to any dynamical properties of the system.
- We quantitatively investigated the intervention of MAS by adding leaders, and provide the proportion of leaders needed for the expected consensus.

Concluding Remarks

- **Analysis of collective behavior**
  - How to analyze other collective behavior, for example, synchronization of the coupled oscillators and the coupled laser array, phase transition of the Vicsek model, flocking behavior of the Boid model?
  - Unified tools and methods to analyze the MAS
  - ……
- **Intervention of MAS**
  - Critical number of leaders to guide the MAS.
  - How to intervene other MAS, such as people in panic, transport systems?
  - How to formulate a theoretical framework for intervention?
  - ……

Thank You!